Algebra Tier 1 Examination

January 2000

1a. Let G be an abelian group of order 60. Consider the homomorphism $\phi : G \to G$ given by $\phi(g) = g^5$. Let $H = \text{Ker}(\phi)$ and $K = \text{Image } (\phi)$. Show that G is the internal direct product of H and K. (Hint: g.c.d. of integers is a linear combination). (8 points)

1b. Would the conclusion in (1a) necessarily be valid for the homomorphism η given by $\eta(g) = g^{10}$? (4 points)

2. A subgroup K of a group G is called a characteristic subgroup if for any automorphism θ of G, $\theta(K) = K$.

   2a. Show that all subgroups of a cyclic group are characteristic. (5 points)

   2b. Show that the center of a group is a characteristic subgroup. (5 points)

   2c. If K is a normal subgroup of G and H is a characteristic subgroup of K show that H is a normal subgroup of G. (7 points)

   2d. Consider the alternating group A_4. Give example of a characteristic subgroup K (of A_4) and a normal subgroup H of K such that H is not a normal subgroup of A_4. (5 points)

3a. Let K be an extension of a field F. If $\alpha \in K$ is transcendental over F, show that so is $\beta = \alpha^2 + \frac{1}{\alpha^2}$. (5 points)

3b. Let K be an extension of F of degree n. Let f be an irreducible polynomial in $F[x]$ of degree m. If the g.c.d. of m and n is 1, show that f remains irreducible when considered as a polynomial in $K[x]$. (Hint: consider a root α of f in an algebraic closure $\overline{F}$ of F which contains K.) (7 points)

4a. Is it possible to have a finite field which is algebraically closed ? Justify your answer. (4 points)

4b. Let E be an extension of $\mathbb{Z}_p$ contained in an algebraic closure $\overline{\mathbb{Z}_p}$. Let f be an irreducible polynomial in $\mathbb{Z}_p[x]$ and let $\alpha, \beta \in \overline{\mathbb{Z}_p}$ be roots of f. If $\alpha \in E$, show that $\beta \in E$. (6 points)

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5. For a ring R with unity 1, an element $r \in R$ is said to be a unit if there exists an element $s \in R$ such that $rs = 1 = sr$.

5a. Find all the units of the ring $\mathbb{Z}[i] = \{m + ni \in \mathbb{C} \mid m, n \in \mathbb{Z}\}$. (Hint: think of modulus of a complex number) (5 points)

5b. Give example of a ring with exactly 20 units. (5 points)

5c. Let $\mathbb{C}[[x]]$ be the ring of formal power series, i.e.

$\mathbb{C}[[x]] = \{\sum_{i=0}^{\infty} a_i x^i \mid a_i \in \mathbb{C}\}$ with addition and multiplication given by

$(\sum_{i=0}^{\infty} a_i x^i) + \sum_{i=0}^{\infty} b_i x^i) = \sum_{i=0}^{\infty}(a_i + b_i)x^i$ and

$(\sum_{i=0}^{\infty} a_i x^i) \cdot (\sum_{i=0}^{\infty} b_i x^i) = \sum_{i=0}^{\infty} c_i x^i$, where $c_i = \sum_{j+k=i} a_j b_k$.

Show that an element $r = \sum_{i=0}^{\infty} a_i x^i$ is a unit in $\mathbb{C}[[x]]$ if and only if $a_0 \neq 0$. (5 points)

5d. Let I be a non-zero ideal in $\mathbb{C}[[x]]$. Show that there exists a positive integer k such that $x^k \in I$. Show further that I is a principal ideal. (7 points)

6a. Let A be an $n \times m$ matrix and let B be $m \times n$ matrix with real coefficients such that $A \cdot B = I_n$, the identity matrix of size n. What is the relationship between n and m ? (Justify your answer). Further, if n = m , show that $B \cdot A = I_n$ as well. (Hint: think of corresponding linear transformations of $\mathbb{R}^n$). (7 points)

6b. Let T be a linear transformation of a finite dimensional vector space over $\mathbb{R}$. Let V_1 (respectively V_{-1}) denote the eigenspace of T for the eigenvalue 1 (respectively -1). If $T^2 = Id$, show that V is a direct sum of V_1 and V_{-1}. (Hint: think of $v + T(v)$). (7 points)

6c. Give an example of a 4×4 matrix with real entries whose real eigenvalues are ± 1 and whose complex eigenvalues are ± 1 and $\pm i$ (no need to justify). (3 points)

6d. Let V be a finite dimensional vector space over $\mathbb{R}$ and W, W' be two subspaces of V. Show that $\dim(W + W') = \dim(W) + \dim(W') - dim(W \cap W')$. (5 points)