Tier 1 Examination - Algebra

August 27, 1999

Justify all answers! All rings are assumed to have an identity element. The set of real numbers is denoted by $\mathbf{R}$.

(8)1. Let G be a group and let H, K be subgroups of G, with H normal. Prove that $HK = \{hk \mid h \in H, k \in K\}$ is a subgroup of G.

(8)2. a. Find the eigenvalues of the following real matrix.

$$\begin{pmatrix} 0 & 3 & 2 \\ 0 & -1 & 0 \\ 1 & 3 & 1 \end{pmatrix}$$

b. Determine whether or not the matrix is diagonalizable over the reals.

(12)3. Let G be a group, H a subgroup of G.
a. Prove or disprove: If the index $[G : H] = 2$, then H is normal.

b. Prove or disprove: If the index $[G : H] = 3$, then H is normal.

(8)4. A ring R is called simple if its only (two-sided) ideals are $\{0\}$ and R. Prove that if R is simple then the center of $R (= \{x \in R \mid xr = rx \text{ for all } r \in R\})$ is a field.

(8)5. Prove that if A and B are real $n \times n$ matrices, then AB and BA have the same eigenvalues.

(7)6. Let p be a prime number and let F be a field of characteristic p. Prove that if $a \in F$ satifies $a^p = 1$, then $a = 1$.

(8)7. Prove that if R is a principal ideal domain, then every nonzero prime ideal is a maximal ideal. Does the conclusion hold in general for R a unique factorization domain?

(8)8. Let F be a finite field. Prove there is a prime p such that the number of elements in F is p^k for some positive integer k.

(15)9. Let V be an n–dimensional real vector space. Let $A(V)$ be the set of linear transformations on V, that is $A(V) = \{T : V \to V \mid T \text{ is a linear transformation}\}$. Recall that $A(V)$ is also a real vector space, if we define the sum by $(S+T)(v) = S(v) + T(v)$ for $S, T \in A(V)$ and $v \in V$ and the scalar product by $(aT)(v) = aT(v)$ for $a \in \mathbf{R}$ and $T \in A(V)$.

a. What is the dimension of $A(V)$?

b. Let $B(V) = \{T \in A(V) \mid \dim T(V) < n\}$. Determine whether or not $B(V)$ is a subspace of $A(V)$. If it is, find its dimension.

c. Let W be a subspace of V of dimension k and let $C(V) = \{T \in A(V) \mid T(w) = 0 \text{ for all } w \in W\}$. Determine whether or not $C(V)$ is a subspace of $A(V)$. If it is, find its dimension.

(10)10. Let $G = Z_3 \times Z_3$, where Z_3 denotes the cyclic group of order 3.
a. Determine the number of distinct homomorphisms from G to itself.

b. Determine the number of distinct isomorphisms from G to itself.

(8)11. Let R be an integral domain and let S be a subring of R. Assume S is a field. Then we may view R as an S–vector space in a natural way. Prove that if R is finite dimensional as an S–vector space, then R is a field.