August 1998

1. Give examples (no need to prove anything): (12 points)

   a. Two non-isomorphic abelen groups of order 108 such that the order of every element divides 72.

   b. A linear transformation $T : \mathbb{C}^3 \to \mathbb{C}^3$ such that the eigenvectors of T do not span $\mathbb{C}^3$.

   c. A unique factorization domain D and a pair of elements $u, v \in D$ such that the greatest common divisor of u and v is NOT a linear combination of u and v.

   d. Three ring homomorphisms from $\mathbb{Z} \to \mathbb{Z}_{10}$.
   (A ring homomorphism between rings A and A' is a map $f : A \to A'$ such that for $a, b \in A$, $f(a+b) = f(a) + f(b)$ and $f(ab) = f(a)f(b)$.)

2. Let G be a group. An equivalence relation $\equiv$ on G is called a congruence relation if $g_1 \equiv g_2$ and $h_1 \equiv h_2$ implies $g_1h_1 \equiv g_2h_2$.

   a. Suppose that $\equiv$ is a congruence relation on G. Show that the equivalence class of the identity element of G is a normal subgroup of G. (5 points)

   b. Let $\phi : G \to G'$ be a homomorphism of groups and $\equiv'$ be a congruence relation on G'. Define a relation $\equiv$ on G by: $x \equiv y$ iff $\phi(x) \equiv' \phi(y)$. Show that $\equiv$ is an equivalence relation on G that is also a congruence relation. (4 points)

3. Let A be a commutative ring with 1. An element $x \in A$ is called nilpotent if $x^r = 0$ for some positive integer r.

   a. Show that the set N of all nilpotent elements in A is an ideal in A and that the quotient ring $A/_N$ has no non-zero nilpotent elements. (6 points)

   b. Show that if $x \in N$, then $1 - x$ is a unit in A. (Hint: Factor $u^r - v^r$ and specialize.) (3 points)

4. Let $\mathbf{P_2}(\mathbb{R})$ be the vector space of all polynomials of degree 2 or less with real coefficients. Consider the linear transformation $T : \mathbf{P_2}(\mathbb{R}) \to \mathbf{P_2}(\mathbb{R})$ given by:

$$T(f)(x) = f(0) + f(1)(x + x^2).$$

Find the eigenvalues of T and determine whether T is diagonalizable. (8 points)

5. Let G be the subgroup of 2×2 complex invertible matrices generated by $x = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$ and $y = \begin{bmatrix} 0 & \omega \\ -\omega^{-1} & 0 \end{bmatrix}$, where ω is a primitive cube-root of 1. Let H and K be the subgroups of G generated by x and y respectively.

a. Show that H and K are normal in G. (6 points)

b. Compute the orders of the subgroups $H, K,\ H \cap K$, and G. (4 points)

6. Let V be an n-dimensional real vector space. Consider the set $F(V)$ of all functions from $V \times V$ to $\mathbb{R}$. It can be proved that $F(V)$ is a real vector space under the operations: For all $\phi, \psi \in F(V)$ and $r \in \mathbb{R}$,

$$(i)\ (\phi + \psi)(x, y) = \phi(x, y) + \psi(x, y), \quad (ii)\ (r\phi)(x, y) = r\phi(x, y).$$

Let $S(V)$ be the subset of all functions $\phi \in F(V)$ which satisfy the following condition: For all $x, y, x' \in V$ and $a, a' \in \mathbb{R}$,

$$(i)\ \phi(x, y) = \phi(y, x), \quad (ii)\ \phi(ax + a'x', y) = a\phi(x, y) + a'\phi(x', y).$$

a. Show that $S(V)$ is a subspace of $F(V)$. (3 points)

b. Find the dimension of $S(V)$. (Hint: Use a suitable map from $S(V)$ to the vector space of all $n \times n$ matrices.) (5 points)

7. Consider the ring $\mathbb{Z}_2[x]$ and two ideals I and J generated by the elements $(x^2 - 1)$ and $x^2 + x + 1$ respectively.

   a. Find all the units in the quotient rings $\mathbb{Z}_2[x]/_I$ and $\mathbb{Z}_2[x]/_J$. (7 points)

   b. If F is a field of 4 elements, is it true that F is isomorphic to one of these two rings? (Justify your answer.) (3 points)

8. Find the irreducible polynomial over $\mathbb{Q}$ of the element $\alpha = \sqrt{5} \cdot \sqrt[3]{2}$. (Hint: Prove first that $\mathbb{Q}(\alpha) = \mathbb{Q}(\sqrt{5}, \sqrt[3]{2})$.) (9 points)

9. Let R be the ring $\mathbb{Z}[i]$ of Gaussian integers (i.e. $R = \{a + bi \mid a, b \in \mathbb{Z}\}$.)

   a. Let $p \in \mathbb{Z}$ be a prime integer. Show that p is a prime element of R if the equation $x^2 + y^2 = p$ has no integer solutions for x and y. (Hint: use the norm $N(a + bi) = a^2 + b^2$.)

   (5 points)

   b. Using (a) or otherwise, show that 11 does not divide $4n^2 + 1$ for all $n \in \mathbb{Z}$. (5 points)

10. Give reasons why the following examples *do not* exist: (15 points)

   a. Elements in $x, y \in S_5$ of order 3 and 4 respectively such that $xy = yx$.

   b. Elements $\alpha, \beta \in \mathbb{C}$ with α transcendental over $\mathbb{Q}$ and $\beta, \alpha^2 + \beta$ both algebraic over $\mathbb{Q}$.

   c. An integral domain with 20 elements.