Tier 1 Examination - Algebra

January , 1998

Note: (1) Justify your answers.
(2) All rings are assumed to have identity elements.
(3)The ring of integers is denoted $\mathbf{Z}$; the real numbers are denoted $\mathbf{R}$.

(24)1. Prove each of the following statements:

(a) If G and H are groups and $f : G \to H$ is a group homomorphism, then the kernel of f is a normal subgroup of G.

(b) If N is a normal subgroup of index 12 in a group G and $g \in G$ with $g^5 \in N$, then $g \in N$.

(c) If R is a commutative ring in which the only ideals are 0 and R, then R is a field.

(d) If I and J are ideals in a commutative ring R then the set $S = \{r \in R | rI \subseteq J\}$ is an ideal of R.

(8)2. Determine the number of subgroups of the group $C_5 \times C_5$.

(8)3. Determine the units in the polynomial ring $\mathbf{F_2}[x]$, where $\mathbf{F_2}$ denotes the ring $\mathbf{Z}/2\mathbf{Z}$.

(10)4. Determine whether the following matrix is diagonalizable over $\mathbf{R}$.

$$\begin{pmatrix} 0 & -1 & -1 \\ 1 & 5 & 4 \\ -1 & -7 & -5 \end{pmatrix}$$

(10)5. Classify the finite abelian groups with the property that every proper subgroup is cyclic. (A subgroup H of a group G is called proper if $H \neq G$.)

(10)6. Let R be a finite ring, not necessarily commutative. Prove that if $a \in R$ is not invertible then there must exist a nonzero element $b \in R$ such that $ab = 0$.

(10)7. Prove that if R is a commutative ring and the polynomial ring $R[x]$ is a PID, then R must be a field.

(10)8. Let V be a finite dimensional real vector space and let $T : V \to V$ be a linear transformation such that $T^2 = T$.

(a) Prove that $ker(T)$ and $T(V)$ are complementary subspaces of V, that is that $ker(T) \cap T(V) = 0$ and $ker(T) + T(V) = V$.

(b) Prove there is a basis of V for which the matrix of T has the following form, where I_m is the $m \times m$ identity matrix and $0_{r,t}$ is the $r \times t$ zero matrix.

$$\begin{pmatrix} I_m & 0_{m,s} \\ 0_{s,m} & 0_s \end{pmatrix}$$

(10)9. Let F be a field and let K/F be a field extension of odd degree. Prove that if $K = F(a)$ for some element $a \in K$, then $F(a) = F(a^2)$.