TIER ONE ALGEBRA EXAM

August, 1996

The ring of integers is denoted $\mathbf{Z}$. The ring of rational numbers is denoted $\mathbf{Q}$. All rings have identity.

(30)1. Give an example of each of the following (No justification required):

(a) A nonabelian group of order 18.

(b) A ring R with exactly four ideals (including R and $\{0\}$).

(c) A nonabelian group all of whose proper subgroups are cyclic.

(d) An element in S_{11} of order 21.

(e) An infinite group all of whose elements are of finite order.

(f) A noncommutative ring in which every nonzero element is either a zero divisor or a unit.

(15)2. Let H, K be subgroups of a group G with K normal.

(a) Prove that $HK = \{hk | h \in H, k \in K\}$ is a subgroup of G.

(b) Provide an example to show that HK need not be a subgroup if neither H nor K is normal.

(15)3. Let G be a group with exactly three subgroups (including $\{e\}$ and G).

(a) Prove that G is a cyclic group.

(b) Prove that the order of G is p^2 for some prime p.

(15)4. Let p be a prime and let $R = \{a \in \mathbf{Q} | a = n/m \text{ for } n, m \in \mathbf{Z}, p \nmid m\}$.

(a) Prove R is a ring under the usual operations in $\mathbf{Q}$.

(b) Prove that R contains a unique maximal ideal.

(10)5. Let R be a commutative ring. Let I, J be ideals in R. Prove that the canonical ring homomorphism $\pi : R \to R/I \oplus R/J$ given by $\pi(r) = (r + I, r + J)$ is an isomorphism if and only if

(1) $I \cap J = 0$

and (2) $I + J = R$.

(10)6. An element x in a ring R is called nilpotent if $x \neq 0$ but $x^k = 0$ for some $k > 0$. Find a necessary and sufficient condition on n for the ring $\mathbf{Z}/n\mathbf{Z}$ to contain a nilpotent element. Prove your answer.

(15)7. (a) Prove that if F is a field then F contains a subfield isomorphic to $\mathbf{Q}$ or to $\mathbf{Z}/p\mathbf{Z}$ for some prime p.

(b) Prove that if F is a finite field then there is a prime q such that the number of elements in F is q^k for some positive integer k.

(c) Prove there exists a field containing exactly 7^3 elements.

(10)8. Let R be an integral domain and let S be a subring of R. Prove that if S is a field and R is finite dimensional as an S–vector space, then R is a field.

(20)9. Let A be the following 3x3 matrix over $\mathbf{Q}$:

$$\begin{pmatrix} 0 & 1 & -2 \\ 0 & 0 & -2 \\ 1 & 0 & 0 \end{pmatrix}$$

(a) Find the characteristic polynomial of A.

(b) Find the minimal polynomial of A over $\mathbf{Q}$.

(c) Let $\mathbf{Q}[A] = \{a_0 I + a_1 A + \ldots + a_k A^k | k \geq 0, a_0, a_1, \ldots a_n \in \mathbf{Q}\}$. Prove $\mathbf{Q}[A]$ is a field.

Tier 1 Algebra Examination

August, 1997.

Time: 3 Hours

Question 1 is worth 25 points, questions 2 through 4 are worth 10 points each and questions 5 through 7 are worth 15 points each.

Start each question on a fresh sheet of paper.

1. Give examples (no need to prove anything) or give mathematical reasons if you can not give examples:

   a. An infinite group all of whose elements have orders 1 or 3.

   b. Matrices A and B of sizes 3 x 2 and 2 x 3 respectively such that $A \cdot B = I_3$, (identity matrix of size 3 x 3)

   c. Two elements in the alternating group A_5 which are conjugate in the symmetric group S_5 but not in A_5.

   d. An u.f.d. which is not a p.i.d.

   e. A transcendental element $\alpha \in \mathbb{C}$ such that $\alpha - \frac{1}{\alpha}$ is an algebraic element.

2. a. Let f be an automorphism of the group $\mathbb{Z}/16\mathbb{Z}$. Show that there exists an odd integer m ($1 \leq m \leq 15$) such that $f(x) = mx$ for all $x \in \mathbb{Z}/16\mathbb{Z}$.

   b. Decompose the group $\mathrm{Aut}(\mathbb{Z}/16\mathbb{Z})$ of all automorphisms of $\mathbb{Z}/16\mathbb{Z}$ as a product of cyclic groups.

3. a. Let G be a group and H be a subgroup. Let $N = \{ g \in G \mid gHg^{-1} = H \}$ be the normalizer of H in G. Show that there is a bijective correspondence between the left cosets of N in G and the set $\mathcal{S} = \{ K \mid K = xHx^{-1} \text{ for some } x \in G \}$ of all conjugates of H.

   b. Let G, H, and N be as in (a) above. If in addition, G is finite with $|H| = r$ and $[G:H] = s$, then show that the union of all members of $\mathcal{S}$ (i.e. $\bigcup_{K \in \mathcal{S}} K$) has at most $(rs - s + 1)$ elements.

4. Let V be an n–dimensional vector space.

a. Show that a proper subspace W of V is the intersection of all subspaces of V of dimension n–1 which contain W.

b. Let A(V) be the vector space of all linear transformations of V to itself. For $x \neq 0$ in V, compute the dimension of $A_x(V) = \{ T \in A(V) \mid T(x) = 0 \}$.

5. Let R be a commutative ring with 1. For an ideal I of R, define $\sqrt{I}$ to be the set $\{ x \in R \mid x^n \in I \text{ for some integer } n \geq 1 \}$.

a. Show that $\sqrt{I}$ is an ideal of R which contains I.

b. If I is a prime ideal, show that $\sqrt{I} = I$.

c. If R is a u.f.d and x is a non–zero, non–unit element in R, find a y such that $\sqrt{R.x} = R.y$. (Hint: consider a prime power factorization for x).

6. Let R be an Euclidean domain with a valuation v (i.e. v is a function from the set of non–zero elements of R to the set of non–negative integers such that (i) $v(x) \leq v(xy)$ for $x, y \in R \setminus \{0\}$ and (ii) given $z \in R$ and $y \in R \setminus \{0\}$, there exist q and r such that $z = yq + r$ with $r = 0$ or $v(r) < v(y)$). Assume further that $v^{-1}(n)$ is finite for all n.

a. Show that for any non–zero ideal I, R/I is finite. (Note that I = R.y for some y).

b. For the ring $\mathbb{Z}[i] = \{ a + bi \in \mathbb{C} \mid a, b \in \mathbb{Z} \}$ of Gaussian integers with standard valuation v given by $v(a + bi) = a^2 + b^2$, prove that $\mathbb{Z}[i] / 3 \cdot \mathbb{Z}[i]$ is a field of 9 elements. (Hint: show that 3 is a prime)).

7. Let p be a prime and n be a positive integer relatively prime to p. Let K be the splitting field of $x^n - 1$ over F_p, the prime field of p elements. Let $[K : F_p] = m$.

a. Show that n divides $p^m - 1$. (a hint is given below)

b. If r is such that n divides $p^r - 1$, show that $m \leq r$.

(Hint for parts (a) and (b): Show first that roots of $x^n - 1$ are all distinct and they form a subgroup of the multiplicative group $K \setminus \{0\}$ which is cyclic).

c. Find $[K : F_3]$ where K is the splitting field of $x^{14} - 1$.