Operational semantics for disintegration

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Mathematical Foundations of Programming Semantics
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What is semantics for?

1. Language for composing generative stories

normal 0 10

(monadic bind, unit)

2. Guarantee that terms denote distributions

(product measure *tricky*)

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3. Equational reasoning by semantics-preserving rewriting!

Fubini

$$\begin{aligned}\textbf{do } \{&x \leftsquigarrow m; \\ &y \leftsquigarrow n; \\ &\textbf{return } (x,y)\}\end{aligned} \quad = \quad \begin{aligned}\textbf{do } \{&y \leftsquigarrow n; \\ &x \leftsquigarrow m; \\ &\textbf{return } (x,y)\}\end{aligned}$$

Conjugacy

$$\begin{aligned}\textbf{do } \{&x \leftsquigarrow \textbf{normal } 0\ 1; \\ &\textbf{factor } e^{-x^2}; \\ &\textbf{return } x\}\end{aligned} \quad = \quad \begin{aligned}\textbf{do } \{&\textbf{factor } (1/\sqrt{3}); \\ &x \leftsquigarrow \textbf{normal } 0\ \sqrt{3}; \\ &\textbf{return } x\}\end{aligned}$$

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(convenient **factor** expresses non-(sub)probability measures)

Two program transformations specified semantically

1. Simplification

$$m = m'$$

2. Disintegration

$$m = \textbf{do}\ \{t \leftsquigarrow m_1; \\ x \leftsquigarrow m_2; \\ \textbf{return}\ (t, x)\}$$

(typically t appears free in m_2)

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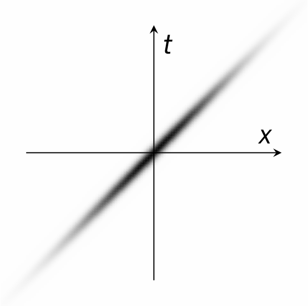
Disintegration defines conditional distributions



$$\begin{array}{ll} \mathbf{do}\ \{x \leftsquigarrow \mathbf{normal}\ 0\ 10; & \text{-- } \textit{latent cause} \\ \quad t \leftsquigarrow \mathbf{normal}\ x\ 1; & \text{-- } \textit{observed effect} \\ \quad \mathbf{return}\ (t, x)\} & \end{array}$$

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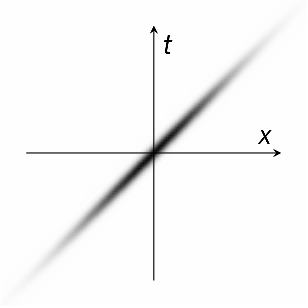

$$\textbf{do}\ \{x \leftsquigarrow \textbf{normal}\ 0\ 10; \quad \text{-- } \textit{latent cause}$$
$$t \leftsquigarrow \textbf{normal}\ x\ 1; \quad \text{-- } \textit{observed effect}$$
$$\textbf{return}\ (t, x)\}$$

$$= \textbf{do}\ \{t \leftsquigarrow \textbf{normal}\ 0\ \sqrt{101};$$
$$x \leftsquigarrow \textbf{normal}\ (t \times 100/101)\ (10/\sqrt{101});$$
$$\textbf{return}\ (t, x)\}$$

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$$m \;=\; \textbf{do}\ \{t \leftsquigarrow m_1;\ x \leftsquigarrow m_2;\ \textbf{return}\ (t,x)\} \;=\; \textbf{do}\ \{t \overset{\star}{\leftsquigarrow} m_1;\ m_2\}$$

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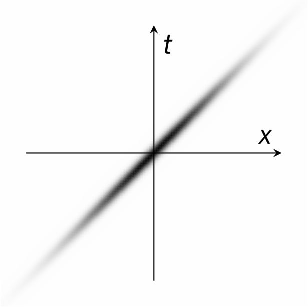

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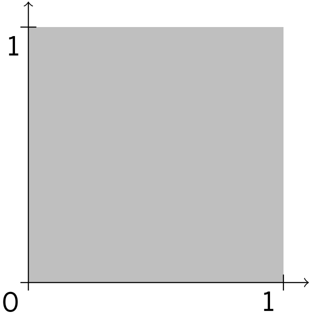
Disintegration allows an uncountable space of observations



$$\textbf{do}\ \{x \leftsquigarrow \textbf{uniform}\ 0\ 1;\ y \leftsquigarrow \textbf{uniform}\ 0\ 1;\ t \leftsquigarrow \qquad ;\ \textbf{return}\ (t, (x, y))\}$$

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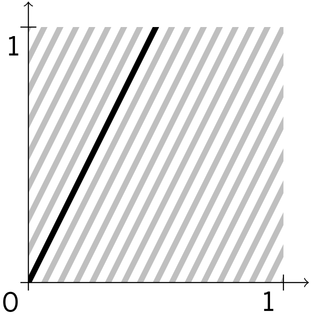

$$\begin{aligned}\textbf{do}\ \{&x \leftsquigarrow \textbf{uniform}\ 0\ 1;\\ &y \leftsquigarrow \textbf{uniform}\ 0\ 1;\\ &t \leftsquigarrow \textbf{return}\ (y - 2 \times x);\\ &\textbf{return}\ (t, (x, y))\}\end{aligned}$$

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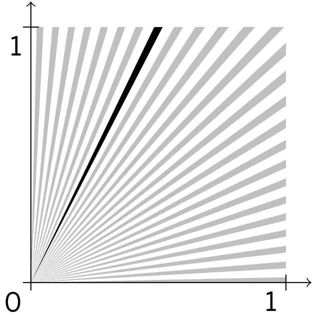


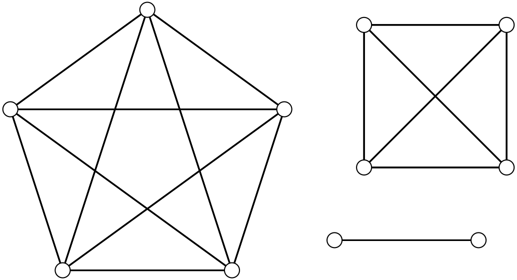
$$\mathbf{do}\ \{x \leftsquigarrow \mathbf{uniform}\ 0\ 1;\ y \leftsquigarrow \mathbf{uniform}\ 0\ 1;\ t \leftsquigarrow \mathbf{return}\ (y/x);\ \mathbf{return}\ (t, (x, y))\}$$

From denotation to operation

Denotation: equivalence relation

Operation: directed graph

So denotation justifies soundness of operation



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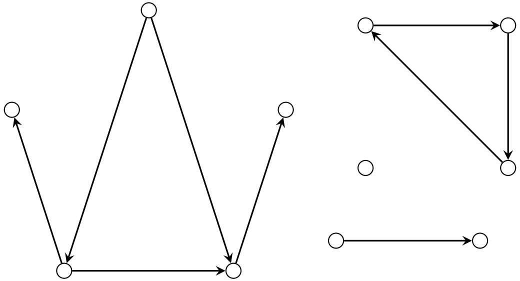

From denotation to operation

Denotation: equivalence relation
Operation: directed graph
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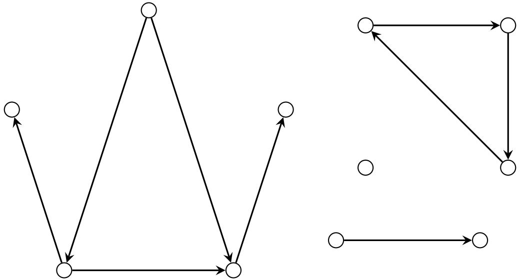

This setup is typical for execution, but what about disintegration?
Rest of this talk: a disintegrator that justifies its work, step by step
A contextual operational semantics, implemented in PLT Redex

Syntax

$$
\begin{aligned}
\text{Term} \quad e, m ::= {} & \textbf{do}\ \{\textbf{factor}\ e;\ m\} \mid \textbf{do}\ \{x \leftsquigarrow m;\ m\} \mid \textbf{do}\ \{x \star\!\!\sim m;\ m\} \\
\mid {} & \textbf{return}\ e \mid \textbf{uniform}\ e\ e \mid \textbf{normal}\ e\ e \mid \textbf{lebesgue} \\
\mid {} & (e, e) \mid \textbf{fst}\ e \mid \textbf{snd}\ e \mid \mathit{number} \mid e - e \mid e/e \mid \cdots \mid x
\end{aligned}
$$

(not shown: conditionals)

Syntax

Stratified syntax during disintegration

State $\quad s ::= \textbf{do}\ \{\textbf{factor}\ e;\ s\} \mid \textbf{do}\ \{x \leftsquigarrow m;\ s\} \mid \textbf{do}\ \{\textbf{delimit};\ t\}$

Thread $\quad t ::= \textbf{do}\ \{\textbf{factor}\ e;\ t\} \mid \textbf{do}\ \{x \leftsquigarrow m;\ t\} \mid \textbf{do}\ \{x \star m;\ m\}$

Term $\quad e, m ::= \textbf{do}\ \{\textbf{factor}\ e;\ m\} \mid \textbf{do}\ \{x \leftsquigarrow m;\ m\}$

$\mid \textbf{return}\ e \mid \textbf{uniform}\ e\ e \mid \textbf{normal}\ e\ e \mid \textbf{lebesgue}$

$\mid (e, e) \mid \textbf{fst}\ e \mid \textbf{snd}\ e \mid \textit{number} \mid e - e \mid e/e \mid \cdots \mid x$

(not shown: conditionals)

Example disintegration

$$\begin{aligned}&\textbf{do } \{x \leftsquigarrow \textbf{uniform } 0\ 1;\\ &\quad y \leftsquigarrow \textbf{uniform } 0\ 1;\\ &\quad \textbf{return } (y/x, (x, y))\}\end{aligned}$$

Stage 1: Isolate (evaluate)

Stage 2: Invert (unevaluate)

Example disintegration

$$
\begin{aligned}
&\textbf{do}\ \{\textbf{delimit};\\
&\quad r \leftsquigarrow \textbf{do}\ \{x \leftsquigarrow \textbf{uniform}\ 0\ 1;\\
&\qquad\qquad\quad y \leftsquigarrow \textbf{uniform}\ 0\ 1;\\
&\qquad\qquad\quad \textbf{return}\ (y/x, (x, y))\};\\
&\quad t \mathrel{\star\!\!\sim} \textbf{return}\ (\textbf{fst}\ r);\\
&\quad \textbf{return}\ (\textbf{snd}\ r)\}
\end{aligned}
$$

Example disintegration (step 1)

```
do {delimit;                                      bind     do {delimit;
    r ⇜ do {x ⇜ uniform 0 1;                      ───→         x ⇜ uniform 0 1;
            y ⇜ uniform 0 1;                                   r ⇜ do {y ⇜ uniform 0 1;
            return (y/x, (x, y))};                                     return (y/x, (x, y))};
    t ⋆⇜ return (fst r);                                       t ⋆⇜ return (fst r);
    return (snd r)}                                            return (snd r)}
```

Example disintegration (step 1)

$$\textbf{do}\ \{\textbf{delimit};\ r \leftsquigarrow \textbf{do}\ \{x \leftsquigarrow \textbf{uniform}\ 0\ 1;\ y \leftsquigarrow \textbf{uniform}\ 0\ 1;\ \textbf{return}\ (y/x, (x, y))\};\ t \star\!\!\leftsquigarrow \textbf{return}\ (\textbf{fst}\ r);\ \textbf{return}\ (\textbf{snd}\ r)\}$$

$$\xrightarrow{\text{bind}}$$

$$\textbf{do}\ \{\textbf{delimit};\ x \leftsquigarrow \textbf{uniform}\ 0\ 1;\ r \leftsquigarrow \textbf{do}\ \{y \leftsquigarrow \textbf{uniform}\ 0\ 1;\ \textbf{return}\ (y/x, (x, y))\};\ t \star\!\!\leftsquigarrow \textbf{return}\ (\textbf{fst}\ r);\ \textbf{return}\ (\textbf{snd}\ r)\}$$

$$\textbf{do}\ \{\textbf{delimit};\ H;\ r \leftsquigarrow \textbf{do}\ \{x \leftsquigarrow m_1;\ m_2\};\ P[r]\}$$

$$\xrightarrow{\text{bind}}$$

$$\textbf{do}\ \{\textbf{delimit};\ H;\ x \leftsquigarrow m_1;\ r \leftsquigarrow m_2;\ P[r]\}$$

H is a heap of bindings

$P[r]$ is a program that demands r

Example disintegration (step 1)

$$
\begin{aligned}
&\mathbf{do}\ \{\mathbf{delimit};\\
&\qquad x \leftsquigarrow \mathbf{uniform}\ 0\ 1;\\
&\qquad r \leftsquigarrow \mathbf{do}\ \{y \leftsquigarrow \mathbf{uniform}\ 0\ 1;\\
&\qquad\qquad\qquad \mathbf{return}\ (y/x, (x,y))\};\\
&\qquad t \overset{\star}{\leftsquigarrow} \mathbf{return}\ (\mathbf{fst}\ r);\\
&\qquad \mathbf{return}\ (\mathbf{snd}\ r)\}
\end{aligned}
$$

Example disintegration (step 2)

$$
\begin{aligned}
&\textbf{do}\ \{\textbf{delimit};\\
&\quad x \leftsquigarrow \textbf{uniform}\ 0\ 1;\\
&\quad r \leftsquigarrow \textbf{do}\ \{y \leftsquigarrow \textbf{uniform}\ 0\ 1;\\
&\qquad\qquad\quad \textbf{return}\ (y/x, (x, y))\};\\
&\quad t \star\!\!\sim \textbf{return}\ (\textbf{fst}\ r);\\
&\quad \textbf{return}\ (\textbf{snd}\ r)\}
\end{aligned}
$$

Example disintegration (step 2)

$$\begin{array}{l}\textbf{do}\ \{\textbf{delimit};\\ \quad x \leftsquigarrow \textbf{uniform}\ 0\ 1;\\ \quad r \leftsquigarrow \textbf{do}\ \{y \leftsquigarrow \textbf{uniform}\ 0\ 1;\\ \qquad\qquad \textbf{return}\ (y/x, (x,y))\};\\ \quad t \star\!\!\leftsquigarrow \textbf{return}\ (\textbf{fst}\ r);\\ \quad \textbf{return}\ (\textbf{snd}\ r)\}\end{array} \xrightarrow{\text{bind}} \begin{array}{l}\textbf{do}\ \{\textbf{delimit};\\ \quad x \leftsquigarrow \textbf{uniform}\ 0\ 1;\\ \quad y \leftsquigarrow \textbf{uniform}\ 0\ 1;\\ \quad r \leftsquigarrow \textbf{return}\ (y/x, (x,y));\\ \quad t \star\!\!\leftsquigarrow \textbf{return}\ (\textbf{fst}\ r);\\ \quad \textbf{return}\ (\textbf{snd}\ r)\}\end{array}$$

$$\begin{array}{l}\textbf{do}\ \{\textbf{delimit};\\ \quad H;\\ \quad r \leftsquigarrow \textbf{do}\ \{x \leftsquigarrow m_1;\\ \qquad\qquad m_2\};\\ \quad P[r]\}\end{array} \xrightarrow{\text{bind}} \begin{array}{l}\textbf{do}\ \{\textbf{delimit};\\ \quad H;\\ \quad x \leftsquigarrow m_1;\\ \quad r \leftsquigarrow m_2;\\ \quad P[r]\}\end{array}$$

Example disintegration (step 3)

$$
\begin{array}{l}
\mathbf{do}\ \{\mathbf{delimit};\\
\quad x \leftsquigarrow \mathbf{uniform}\ 0\ 1;\\
\quad y \leftsquigarrow \mathbf{uniform}\ 0\ 1;\\
\quad r \leftsquigarrow \mathbf{return}\ (y/x, (x, y));\\
\quad t \mathrel{\star\!\!\leftsquigarrow} \mathbf{return}\ (\mathbf{fst}\ r);\\
\quad \mathbf{return}\ (\mathbf{snd}\ r)\}
\end{array}
\xrightarrow{\text{return}}
\begin{array}{l}
\mathbf{do}\ \{\mathbf{delimit};\\
\quad x \leftsquigarrow \mathbf{uniform}\ 0\ 1;\\
\quad y \leftsquigarrow \mathbf{uniform}\ 0\ 1;\\
\quad r \leftsquigarrow \mathbf{return}\ (y/x, (x, y));\\
\quad t \mathrel{\star\!\!\leftsquigarrow} \mathbf{return}\ (\mathbf{fst}\ (y/x, (x, y)));\\
\quad \mathbf{return}\ (\mathbf{snd}\ r)\}
\end{array}
$$

$$
\begin{array}{l}
\mathbf{do}\ \{\mathbf{delimit};\\
\quad H;\\
\quad r \leftsquigarrow \mathbf{return}\ v;\\
\quad P[r]\}
\end{array}
\xrightarrow{\text{return}}
\begin{array}{l}
\mathbf{do}\ \{\mathbf{delimit};\\
\quad H;\\
\quad r \leftsquigarrow \mathbf{return}\ v;\\
\quad P[v]\}
\end{array}
$$

v is head normal form for lazy evaluation

Example disintegration (step 4)

$$\mathbf{do}\ \{\mathbf{delimit};\ x \leftsquigarrow \mathbf{uniform}\ 0\ 1;\ y \leftsquigarrow \mathbf{uniform}\ 0\ 1;\ r \leftsquigarrow \mathbf{return}\ (y/x, (x, y));\ t \star\!\!\sim \mathbf{return}\ (\mathbf{fst}\ (y/x, (x, y)));\ \mathbf{return}\ (\mathbf{snd}\ r)\}$$

$$\xrightarrow{\text{fst}}\ \mathbf{do}\ \{\mathbf{delimit};\ x \leftsquigarrow \mathbf{uniform}\ 0\ 1;\ y \leftsquigarrow \mathbf{uniform}\ 0\ 1;\ r \leftsquigarrow \mathbf{return}\ (y/x, (x, y));\ t \star\!\!\sim \mathbf{return}\ (y/x);\ \mathbf{return}\ (\mathbf{snd}\ r)\}$$

$$\mathbf{do}\ \{\mathbf{delimit};\ P[\mathbf{fst}\ (e_1, e_2)]\}\ \xrightarrow{\text{fst}}\ \mathbf{do}\ \{\mathbf{delimit};\ P[e_1]\}$$

Example disintegration (step 5)

$$\mathbf{do}\ \{\ \mathbf{delimit};\ x \leftsquigarrow \mathbf{uniform}\ 0\ 1;\ y \leftsquigarrow \mathbf{uniform}\ 0\ 1;\ r \leftsquigarrow \mathbf{return}\ (y/x, (x, y));\ t \star\!\!\sim \mathbf{return}\ (y/x);\ \mathbf{return}\ (\mathbf{snd}\ r)\} \xrightarrow{\text{Fubini}} \mathbf{do}\ \{x' \leftsquigarrow \mathbf{uniform}\ 0\ 1;\ \mathbf{delimit};\ x \leftsquigarrow \mathbf{return}\ x';\ y \leftsquigarrow \mathbf{uniform}\ 0\ 1;\ r \leftsquigarrow \mathbf{return}\ (y/x, (x, y));\ t \star\!\!\sim \mathbf{return}\ (y/x);\ \mathbf{return}\ (\mathbf{snd}\ r)\}$$

$$\mathbf{do}\ \{\ \mathbf{delimit};\ H;\ x \leftsquigarrow \mathbf{uniform}\ v_1\ v_2;\ P[x]\} \xrightarrow{\text{Fubini}} \mathbf{do}\ \{x' \leftsquigarrow \mathbf{uniform}\ v_1\ v_2;\ \mathbf{delimit};\ H;\ x \leftsquigarrow \mathbf{return}\ x';\ P[x]\}$$

Example disintegration (step 6)

$$\begin{aligned}&\textbf{do}\ \{x' \leftsquigarrow \textbf{uniform}\ 0\ 1;\\ &\quad \textbf{delimit};\\ &\quad x \leftsquigarrow \textbf{return}\ x';\\ &\quad y \leftsquigarrow \textbf{uniform}\ 0\ 1;\\ &\quad r \leftsquigarrow \textbf{return}\ (y/x,(x,y));\\ &\quad t \star\!\!\sim \textbf{return}\ (y/x);\\ &\quad \textbf{return}\ (\textbf{snd}\ r)\}\end{aligned} \xrightarrow{\text{return}} \begin{aligned}&\textbf{do}\ \{x' \leftsquigarrow \textbf{uniform}\ 0\ 1;\\ &\quad \textbf{delimit};\\ &\quad x \leftsquigarrow \textbf{return}\ x';\\ &\quad y \leftsquigarrow \textbf{uniform}\ 0\ 1;\\ &\quad r \leftsquigarrow \textbf{return}\ (y/x,(x,y));\\ &\quad t \star\!\!\sim \textbf{return}\ (y/x');\\ &\quad \textbf{return}\ (\textbf{snd}\ r)\}\end{aligned}$$

$$\begin{aligned}&\textbf{do}\ \{\textbf{delimit};\\ &\quad H;\\ &\quad r \leftsquigarrow \textbf{return}\ v;\\ &\quad P[r]\}\end{aligned} \xrightarrow{\text{return}} \begin{aligned}&\textbf{do}\ \{\textbf{delimit};\\ &\quad H;\\ &\quad r \leftsquigarrow \textbf{return}\ v;\\ &\quad P[v]\}\end{aligned}$$

Example disintegration (step 7)

$$
\begin{array}{l}
\textbf{do}\ \{x' \leftsquigarrow \textbf{uniform}\ 0\ 1;\\
\quad \textbf{delimit};\\
\quad x \leftsquigarrow \textbf{return}\ x';\\
\quad y \leftsquigarrow \textbf{uniform}\ 0\ 1;\\
\\
\quad r \leftsquigarrow \textbf{return}\ (y/x,(x,y));\\
\quad t \mathrel{\star\!\!\sim} \textbf{return}\ (y/x');\\
\quad \textbf{return}\ (\textbf{snd}\ r)\}
\end{array}
\xrightarrow{\text{un}/}
\begin{array}{l}
\textbf{do}\ \{x' \leftsquigarrow \textbf{uniform}\ 0\ 1;\\
\quad \textbf{delimit};\\
\quad z \leftsquigarrow \textbf{uniform}\ (0/x')\ (1/x');\\
\quad y \leftsquigarrow \textbf{return}\ (z \times x');\\
\quad x \leftsquigarrow \textbf{return}\ x';\\
\quad r \leftsquigarrow \textbf{return}\ (y/x,(x,y));\\
\quad t \mathrel{\star\!\!\sim} \textbf{return}\ (z);\\
\quad \textbf{return}\ (\textbf{snd}\ r)\}
\end{array}
$$

$$
\begin{array}{l}
\textbf{do}\ \{\textbf{delimit};\\
\quad H_1;\\
\quad y \leftsquigarrow \textbf{uniform}\ v_1\ v_2;\\
\\
\quad H_2;\\
\quad t \mathrel{\star\!\!\sim} \textbf{return}\ E[y/v];\\
\quad m\}
\end{array}
\xrightarrow{\text{un}/}
\begin{array}{l}
\textbf{do}\ \{\textbf{delimit};\\
\quad z \leftsquigarrow \textbf{uniform}\ (v_1/v)\ (v_2/v);\\
\quad y \leftsquigarrow \textbf{return}\ (z \times v);\\
\quad H_1;\\
\quad H_2;\\
\quad t \mathrel{\star\!\!\sim} \textbf{return}\ E[z];\\
\quad m\}
\end{array}
$$

Example disintegration (step 8)

$$
\begin{array}{l}
\textbf{do}\ \{x' \leftsquigarrow \textbf{uniform}\ 0\ 1; \\
\quad \textbf{delimit}; \\
\quad z \leftsquigarrow \textbf{uniform}\ (0/x')\ (1/x'); \\
\quad y \leftsquigarrow \textbf{return}\ (z \times x'); \\
\quad x \leftsquigarrow \textbf{return}\ x'; \\
\quad r \leftsquigarrow \textbf{return}\ (y/x, (x,y)); \\
\quad t \mathrel{\star\!\!\leftsquigarrow} \textbf{return}\ z; \\
\quad \textbf{return}\ (\textbf{snd}\ r)\}
\end{array}
\xrightarrow{\text{unreturn}}
\begin{array}{l}
\textbf{do}\ \{x' \leftsquigarrow \textbf{uniform}\ 0\ 1; \\
\quad \textbf{delimit}; \\
\quad t \mathrel{\star\!\!\leftsquigarrow} \textbf{uniform}\ (0/x')\ (1/x'); \\
\quad z \leftsquigarrow \textbf{return}\ t; \\
\quad y \leftsquigarrow \textbf{return}\ (z \times x'); \\
\quad x \leftsquigarrow \textbf{return}\ x'; \\
\quad r \leftsquigarrow \textbf{return}\ (y/x, (x,y)); \\
\quad \textbf{return}\ (\textbf{snd}\ r)\}
\end{array}
$$

$$
\begin{array}{l}
\textbf{do}\ \{\textbf{delimit}; \\
\quad H_1; \\
\quad z \leftsquigarrow m_1; \\
\quad H_2; \\
\quad t \mathrel{\star\!\!\leftsquigarrow} \textbf{return}\ z; \\
\quad m_2\}
\end{array}
\xrightarrow{\text{unreturn}}
\begin{array}{l}
\textbf{do}\ \{\textbf{delimit}; \\
\quad H_1; \\
\quad t \mathrel{\star\!\!\leftsquigarrow} m_1; \\
\quad z \leftsquigarrow \textbf{return}\ t; \\
\quad H_2; \\
\quad m_2\}
\end{array}
$$

Example disintegration (step 9)

$$
\begin{array}{l}
\textbf{do}\ \{ \\
\quad x' \leftsquigarrow \textbf{uniform}\ 0\ 1; \\
\quad \textbf{delimit}; \\
\quad t \star\!\!\sim \textbf{uniform}\ (0/x')\ (1/x'); \\
\quad z \leftsquigarrow \textbf{return}\ t; \\
\quad y \leftsquigarrow \textbf{return}\ (z \times x'); \\
\quad x \leftsquigarrow \textbf{return}\ x'; \\
\quad r \leftsquigarrow \textbf{return}\ (y/x, (x, y)); \\
\quad \textbf{return}\ (\textbf{snd}\ r)\}
\end{array}
\xrightarrow{\text{done}}
\begin{array}{l}
\textbf{do}\ \{t \star\!\!\sim \textbf{lebesgue}; \\
\quad x' \leftsquigarrow \textbf{uniform}\ 0\ 1; \\
\quad \textbf{factor}\ \text{if}\ t\ \text{is between}\ 0/x'\ \text{and}\ 1/x' \\
\qquad\qquad \text{then}\ |x|\ \text{else}\ 0; \\
\quad z \leftsquigarrow \textbf{return}\ t; \\
\quad y \leftsquigarrow \textbf{return}\ (z \times x'); \\
\quad x \leftsquigarrow \textbf{return}\ x'; \\
\quad r \leftsquigarrow \textbf{return}\ (y/x, (x, y)); \\
\quad \textbf{return}\ (\textbf{snd}\ r)\}
\end{array}
$$

Summary

Disintegration is useful:

- defines conditional distributions
- allows an uncountable space of observations

Disintegrator is useful:

- generates steps by operational semantics
- justifies steps by denotational semantics
- resembles lazy evaluator + change of variables
- leaves verification conditions behind for exchanging integrals