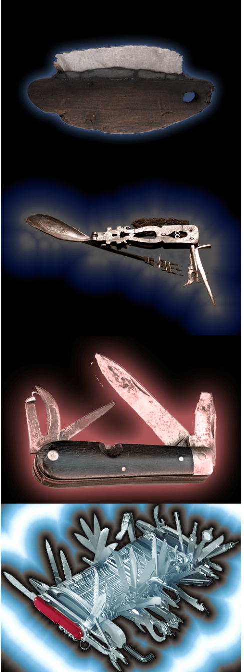
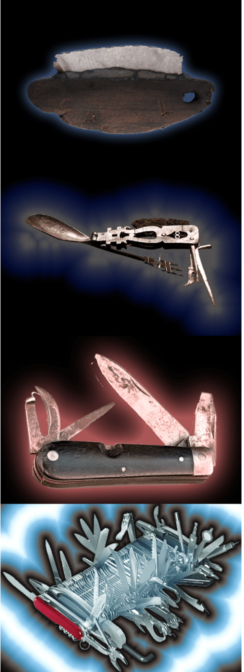
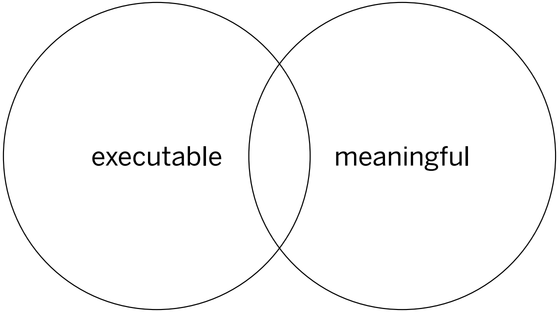


Calculating distributions

Chung-chieh Shan
Indiana University
2018-09-21

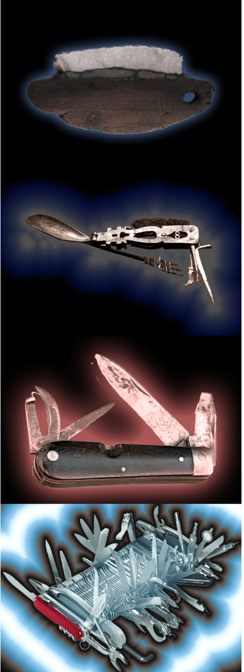


Calculating distributions

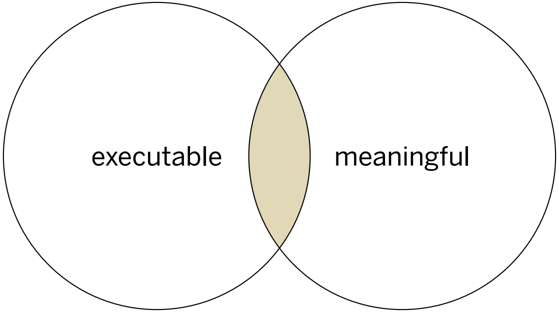


executable

meaningful



Calculating distributions



executable

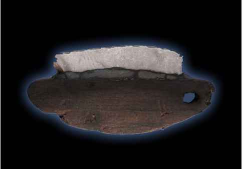
meaningful

I’d also like to address this concept of being “fake” or “calculating.”

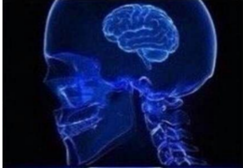
If being “fake” means not thinking or feeling the same way in one moment than you thought or felt in a different moment, then lord help us all.

If being “calculating” is thinking through your words and actions and modeling the behavior you would like to see in the world, even when it is difficult, then I hope more of you will become calculating.

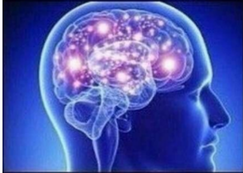
—BenDeLaCreme



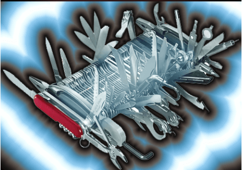
Creative definitions and reasoning from first principles



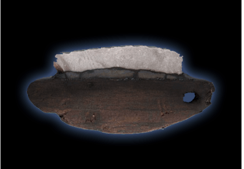
Symbolic representations of common definition patterns



Mechanical operations for common reasoning patterns

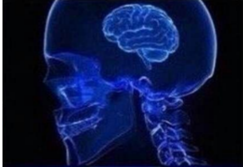


Virtuous cycle of automation and exploration (Buchberger)



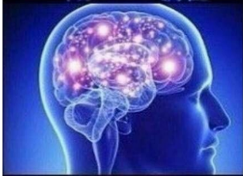
Creative definitions and reasoning from first principles

natural numbers



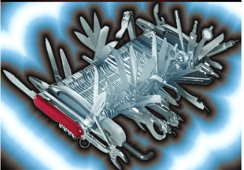
Symbolic representations of common definition patterns

unary, binary



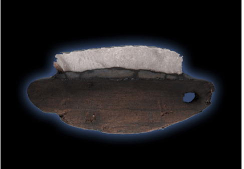
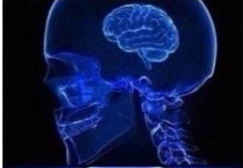
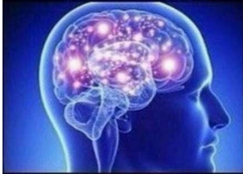
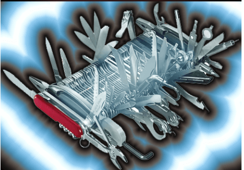
Mechanical operations for common reasoning patterns

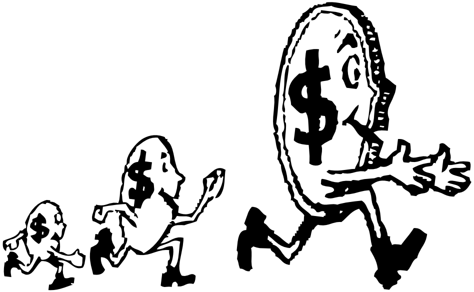
$<$, $+$, $\div$



Virtuous cycle of automation and exploration (Buchberger)

rationals, reals, polynomials

	**Creative definitions and reasoning from first principles**	natural numbers	probability distributions	
	Symbolic representations of common definition patterns	unary, binary	table, Bayes net, probabilistic program	
	Mechanical operations for common reasoning patterns	$<, +, \div$	recognize, integrate, disintegrate	
	**Virtuous cycle of automation and exploration** (Buchberger)	rationals, reals, polynomials	inference, learning, optimization	



An unknown random process yields a stateless coin that can be flipped repeatedly to produce heads (H) or tails (T).

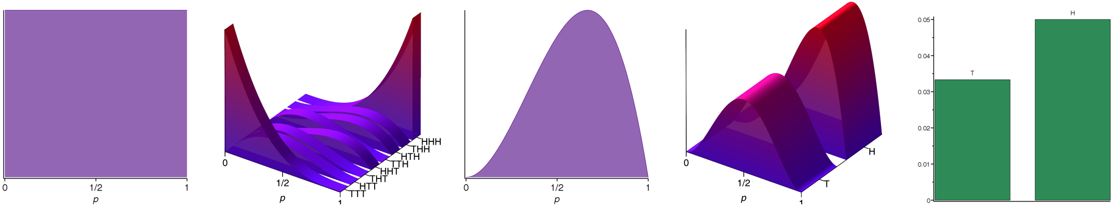
We assume that the probability *p* that the coin produces H each time is distributed uniformly between 0 and 1 by the process.

We flip the coin 3 times and observe THH.

What is the probability that the next flip produces H versus T?

(adapted from Eddy)

$$\begin{tikzcd}
\Pr(p) \arrow[r, bend left, "\text{bind}"] & \Pr(p, \vec{x}) \arrow[r, bend left, "\text{disintegrate}"] & \Pr(p \mid \vec{x}) \arrow[r, bend left, "\text{bind}"] & \Pr(p, y \mid \vec{x}) \arrow[r, bend left, "\text{integrate}"] & \Pr(y \mid \vec{x})
\end{tikzcd}$$



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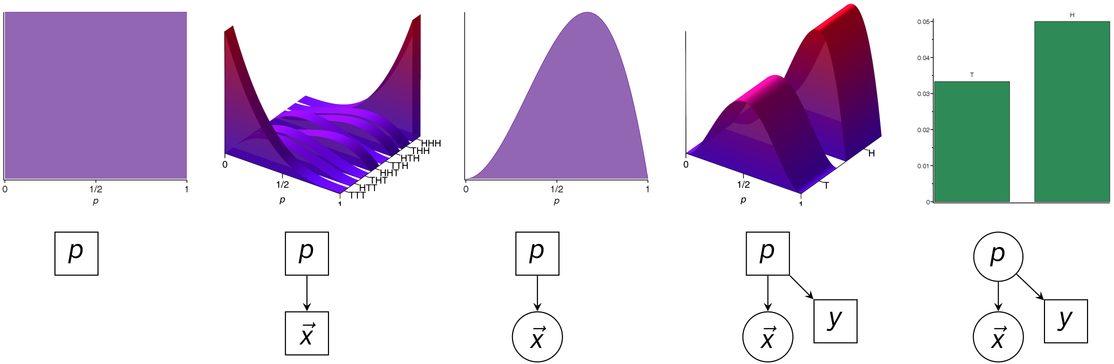

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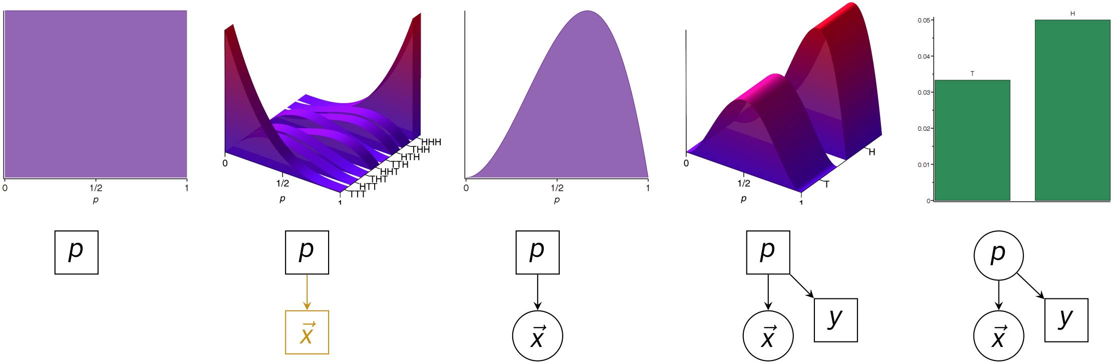

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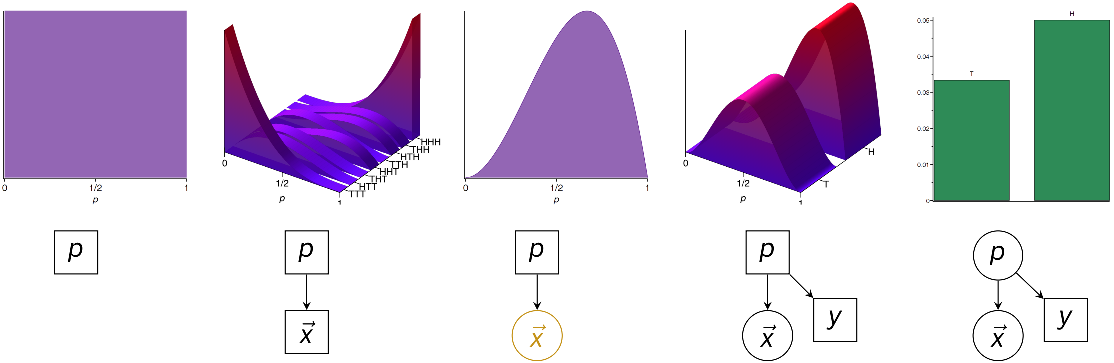

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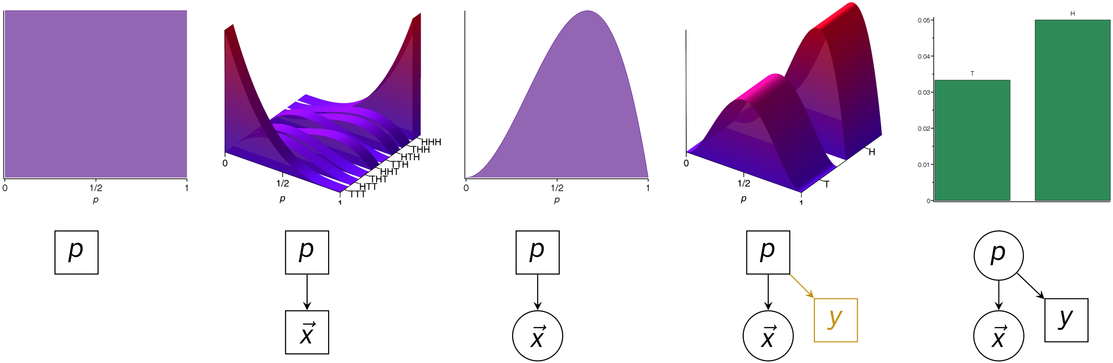

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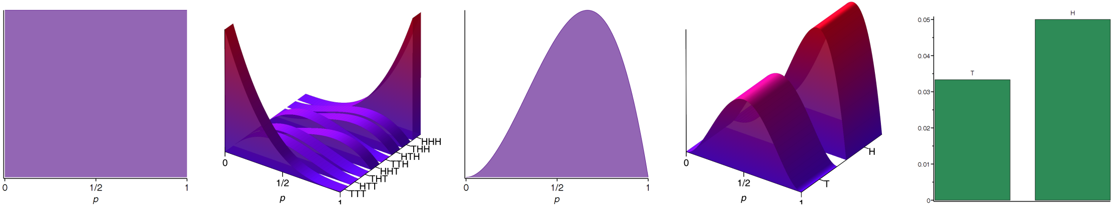

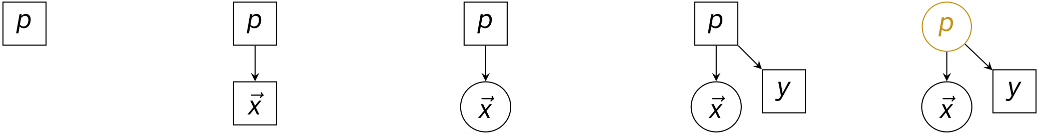

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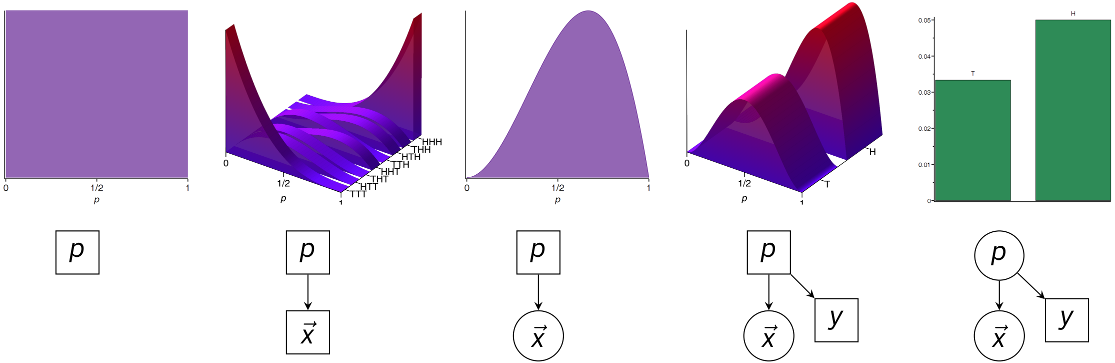

|| || ||

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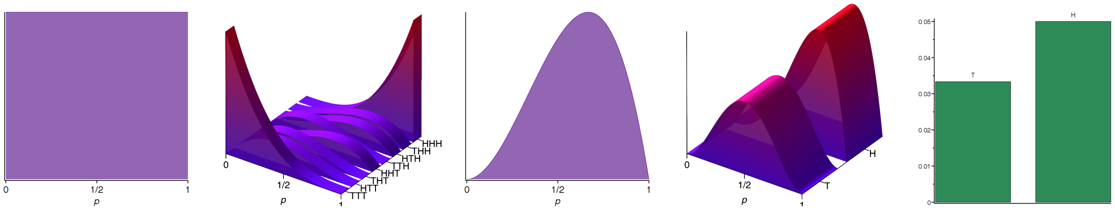
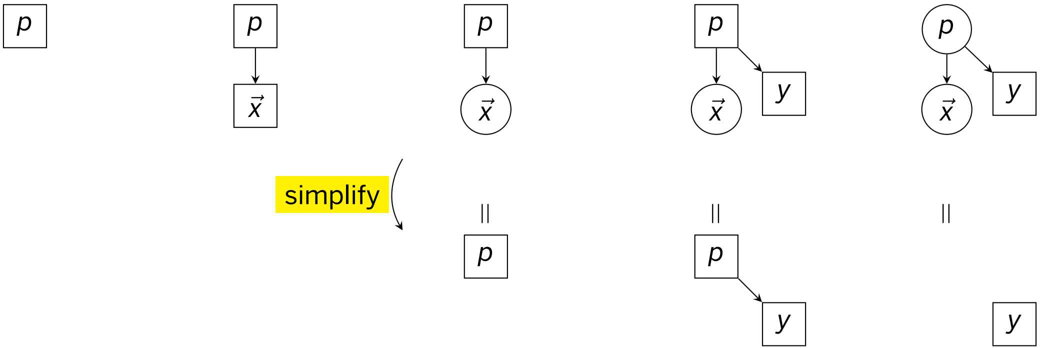
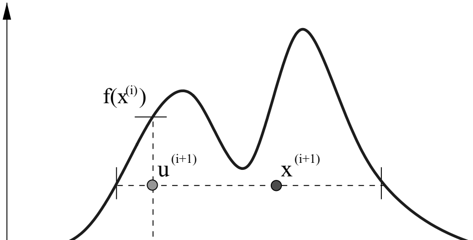




Approximations calculated exactly

Approximations calculated exactly

$$
\begin{aligned}
x_1^{(t+1)} &\sim P(x_1|x_2^{(t)}, x_3^{(t)}, \ldots x_K^{(t)}) \\
x_2^{(t+1)} &\sim P(x_2|x_1^{(t+1)}, x_3^{(t)}, \ldots x_K^{(t)}) \\
x_3^{(t+1)} &\sim P(x_3|x_1^{(t+1)}, x_2^{(t+1)}, \ldots x_K^{(t)}), \text{etc.}
\end{aligned}
$$



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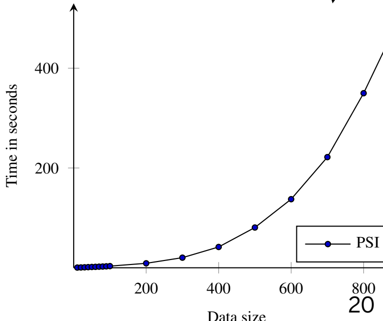
 & & & & {}
\end{tikzcd}$$



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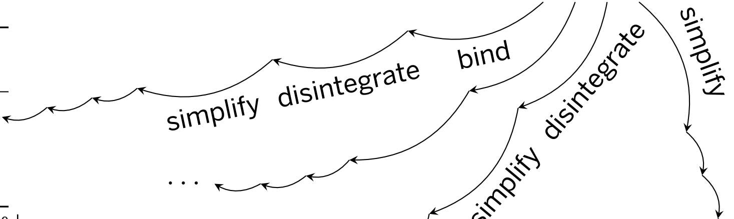
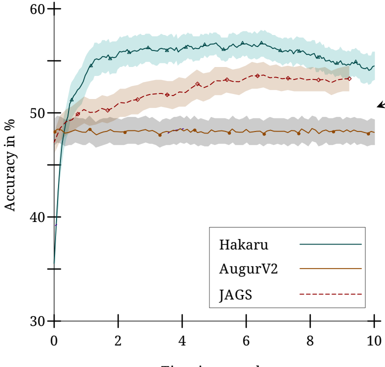
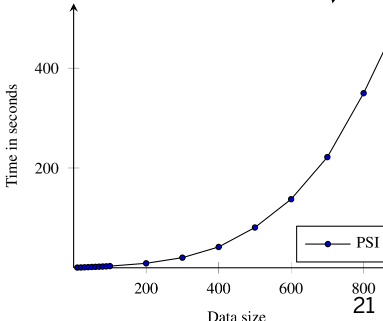


Inference method	Run time (msecs)	
	Mean	SD
WebPPL	1078	16
Hakaru without simplifications	1321	93
Hakaru with simplifications	269	10
Handwritten	207	4

Put approximations in the language!

(FLOPS 2016, UAI 2017)





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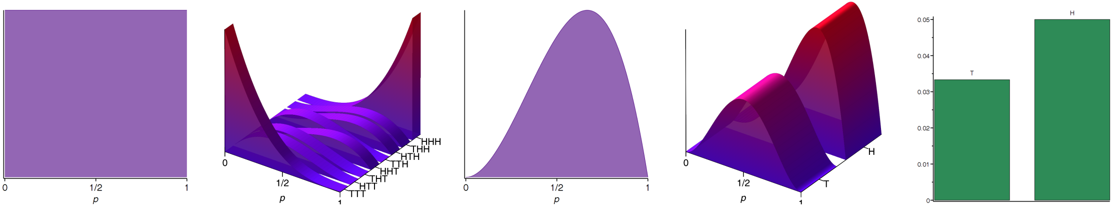
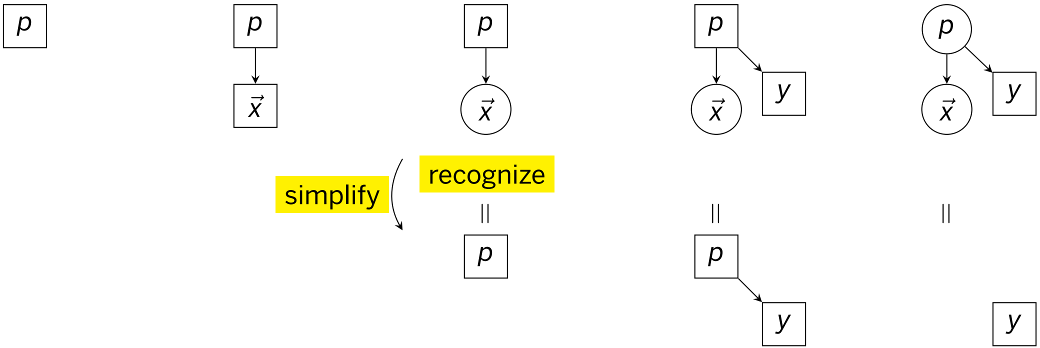


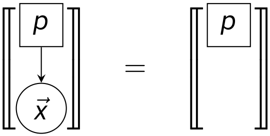


simplify

recognize

Recognizing a density function

Program denote measures:



Recognizing a density function

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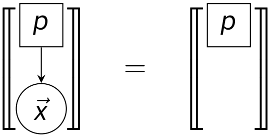

$\in \quad \mathbb{M}\,[0,1]$

Recognizing a density function

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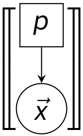

$$= [\![\boxed{p}]\!] \in \mathbb{M}\,[0,1] \subseteq ([0,1] \to \mathbb{R}_+) \to \mathbb{R}_+$$

Measures compute expectations:

Recognizing a density function

Program denote measures:

$$\left[\!\!\left[\begin{array}{c}\boxed{p}\\ \downarrow\\ \textcircled{$\vec{x}$}\end{array}\right]\!\!\right] = \left[\!\!\left[\begin{array}{c}\boxed{p}\\ \\ \\ \end{array}\right]\!\!\right] \in \mathbb{M}\,[0,1] \subseteq ([0,1] \to \mathbb{R}_+) \to \mathbb{R}_+$$

Measures compute expectations:

$$f \quad \mapsto$$

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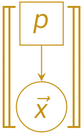

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Measures compute expectations:

$$f \mapsto \int_0^1 \sum_{\vec{x}\in\{\mathrm{H},\mathrm{T}\}^3} p^{\sum_i \langle x_i = \mathrm{H}\rangle} (1-p)^{\sum_i \langle x_1 = \mathrm{T}\rangle} \langle \vec{x} = \mathrm{THH}\rangle \cdot f(p)\,dp$$

Recognizing a density function

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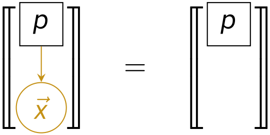

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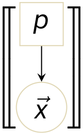

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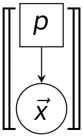
$$= \int_0^1 p^2(1-p) \cdot f(p)\,dp$$

Recognizing a density function

Program denote measures:



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Measures compute expectations:

$$f \mapsto \int_0^1 \sum_{\vec{x} \in \{\mathrm{H},\mathrm{T}\}^3} p^{\sum_i \langle x_i = \mathrm{H} \rangle} (1-p)^{\sum_i \langle x_1 = \mathrm{T} \rangle} \langle \vec{x} = \mathrm{THH} \rangle \cdot f(p)\, dp$$

$$= \int_0^1 p^2 (1-p) \cdot f(p)\, dp$$

Need to recognize simplified denotation as Beta distribution ...

Recognizing a density function

Goal: recognize $h(p) = p^2(1-p)$ as the *density* of **beta** 3 2

Robustness challenge: many equivalent ways to write $p^2(1-p)$ arise

Modularity challenge: many distribution families (**beta**, **normal**, ...) known

Recognizing a density function

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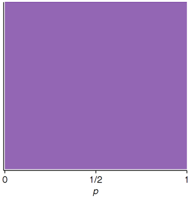
Modularity challenge: many distribution families (**beta**, **normal**, ...) known

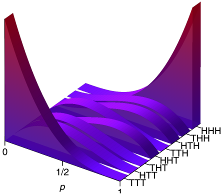
Solution: characterize density functions by their **holonomic representation**, a *homogeneous linear differential equation* such as

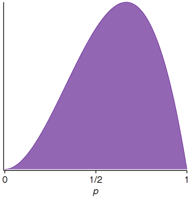
$$p(1-p) \cdot h'(p) + \big(p - 2(1-p)\big) \cdot h(p) = 0$$

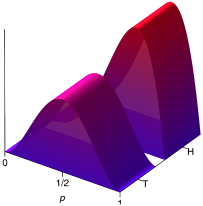
computed *compositionally*!

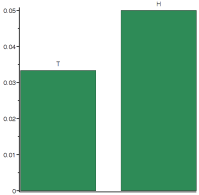
$$\begin{tikzcd}
\Pr(p) \arrow[r, bend left, "\text{bind}"] & \Pr(p, \vec{x}) \arrow[r, bend left, "\text{disintegrate}"] & \Pr(p \mid \vec{x}) \arrow[r, bend left, "\text{bind}"] & \Pr(p, y \mid \vec{x}) \arrow[r, bend left, "\text{integrate}"] & \Pr(y \mid \vec{x})
\end{tikzcd}$$

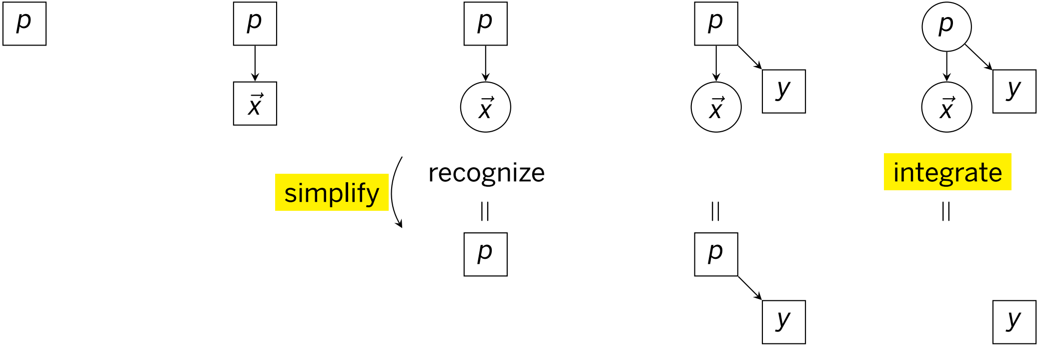












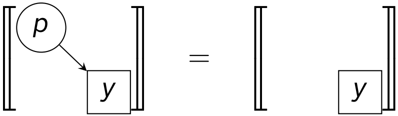
simplify

recognize

integrate

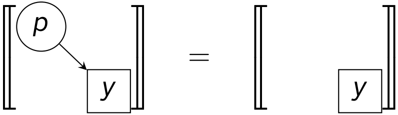
Eliminating a random variable

Program denote measures:



Eliminating a random variable

Program denote measures:



$$\in \quad \mathbb{M}\,\{\mathsf{H},\mathsf{T}\} \quad \subseteq \quad (\{\mathsf{H},\mathsf{T}\} \to \mathbb{R}_+) \to \mathbb{R}_+$$

Measures compute expectations:

$$f \quad \mapsto \quad \int_0^1 p^2(1-p) \sum_{y\in\{\mathsf{H},\mathsf{T}\}} p^{\langle y=\mathsf{H}\rangle}(1-p)^{\langle y=\mathsf{T}\rangle} \cdot f(y)\,dp$$

Eliminating a random variable

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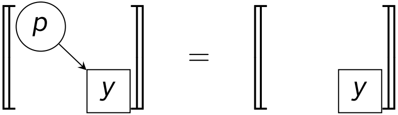

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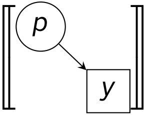
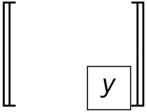


$$=$$



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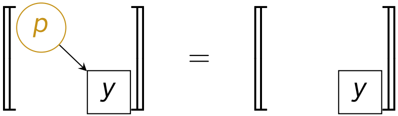

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Eliminating a random variable

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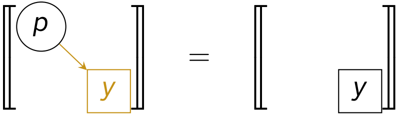

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Eliminating a random variable

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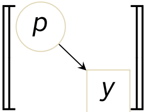
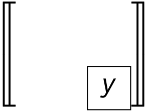


$$=$$



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Measures compute expectations:

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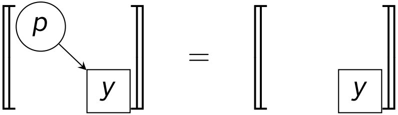
Eliminating a random variable

Program denote measures:



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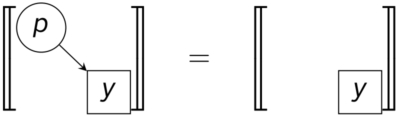
$$= \quad \sum_{y\in\{\mathrm{H},\mathrm{T}\}} \left(\int_0^1 p^2(1-p)p^{\langle y=\mathrm{H}\rangle}(1-p)^{\langle y=\mathrm{T}\rangle}\, dp \right) \cdot f(y)$$

Eliminating a random variable

Program denote measures:



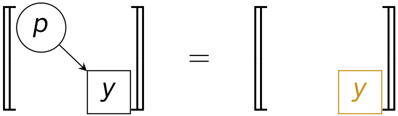
$$\in \quad \mathbb{M}\,\{\mathrm{H},\mathrm{T}\} \quad \subseteq \quad (\{\mathrm{H},\mathrm{T}\} \to \mathbb{R}_+) \to \mathbb{R}_+$$

Measures compute expectations:

$$\begin{aligned}
f \quad \mapsto \quad & \int_0^1 p^2(1-p) \sum_{y\in\{\mathrm{H},\mathrm{T}\}} p^{\langle y=\mathrm{H}\rangle}(1-p)^{\langle y=\mathrm{T}\rangle} \cdot f(y)\,dp \\
= \quad & \sum_{y\in\{\mathrm{H},\mathrm{T}\}} \left(\int_0^1 p^2(1-p)p^{\langle y=\mathrm{H}\rangle}(1-p)^{\langle y=\mathrm{T}\rangle}\,dp \right) \cdot f(y) \\
= \quad & \sum_{y\in\{\mathrm{H},\mathrm{T}\}} \left(\frac{1}{20}\right)^{\langle y=\mathrm{H}\rangle} \left(\frac{1}{30}\right)^{\langle y=\mathrm{T}\rangle} \cdot f(y)
\end{aligned}$$

Eliminating a random variable

Program denote measures:



$$\in \quad \mathbb{M}\,\{\mathsf{H},\mathsf{T}\} \quad \subseteq \quad (\{\mathsf{H},\mathsf{T}\} \to \mathbb{R}_+) \to \mathbb{R}_+$$

Measures compute expectations:

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$$\begin{tikzcd}
\Pr(p) \arrow[r, bend left, "\text{bind}"] & \Pr(p,\vec{x}) \arrow[r, bend left, "\text{disintegrate}"] & \Pr(p \mid \vec{x}) \arrow[r, bend left, "\text{bind}"] & \Pr(p,y \mid \vec{x}) \arrow[r, bend left, "\text{integrate}"] & \Pr(y \mid \vec{x})
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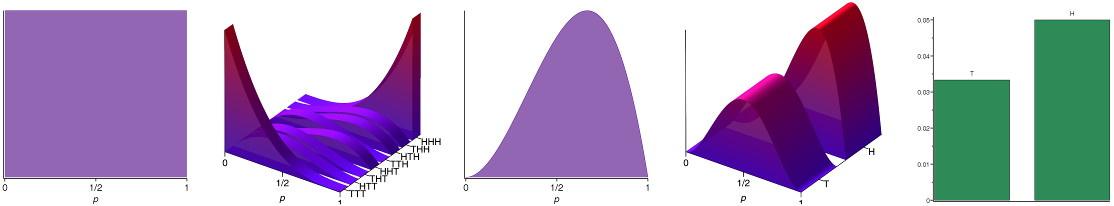
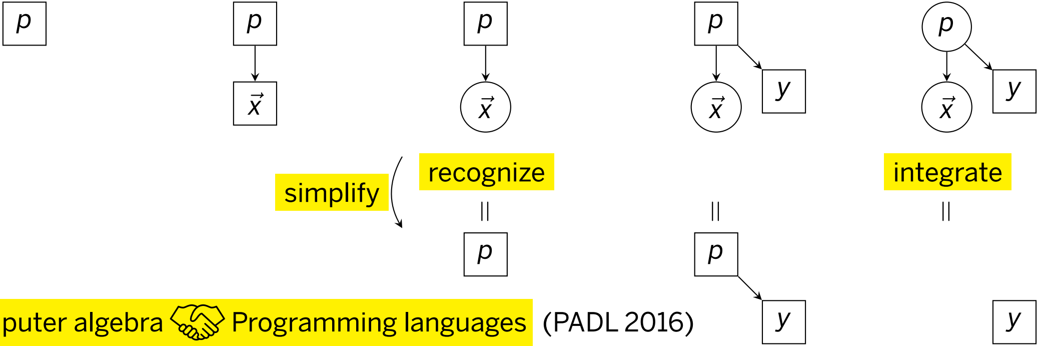
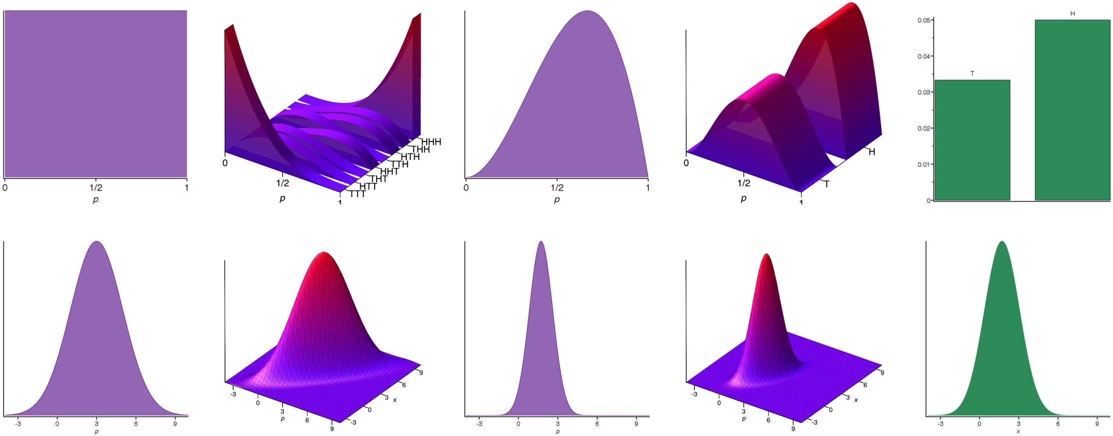




Computer algebra 🤝 Programming languages (PADL 2016)

$$\begin{tikzcd}
\Pr(p) \arrow[r, bend left, "\text{bind}"] & \Pr(p, \vec{x}) \arrow[r, bend left, "\text{disintegrate}"] & \Pr(p \mid \vec{x}) \arrow[r, bend left, "\text{bind}"] & \Pr(p, y \mid \vec{x}) \arrow[r, bend left, "\text{integrate}"] & \Pr(y \mid \vec{x})
\end{tikzcd}$$



$$\begin{tikzcd}
\Pr(p) \arrow[r, bend left, "\text{bind}"] & \Pr(p, \vec{x}) \arrow[r, bend left, "\text{disintegrate}"] & \Pr(p \mid \vec{x}) \arrow[r, bend left, "\text{bind}"] & \Pr(p, y \mid \vec{x}) \arrow[r, bend left, "\text{integrate}"] & \Pr(y \mid \vec{x})
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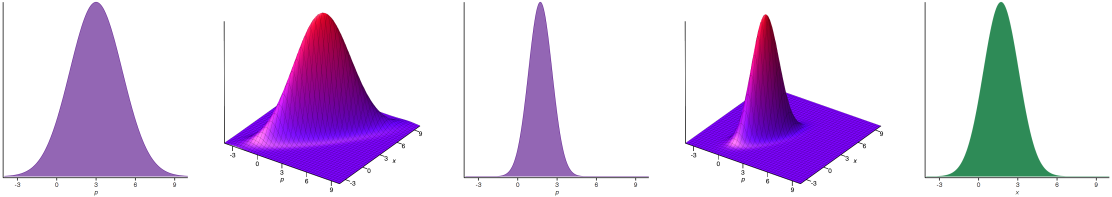


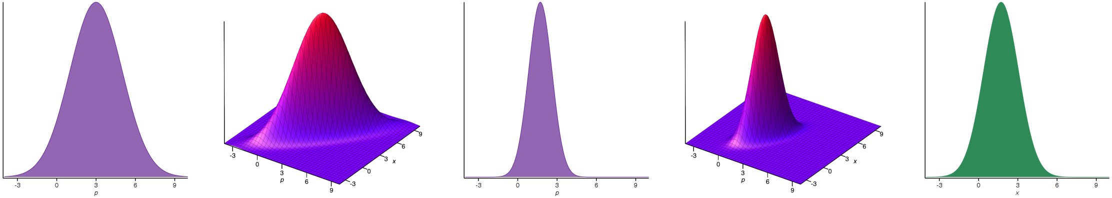
An unknown random process yields a stateless particle whose one-dimensional position can be measured repeatedly to produce a real number.

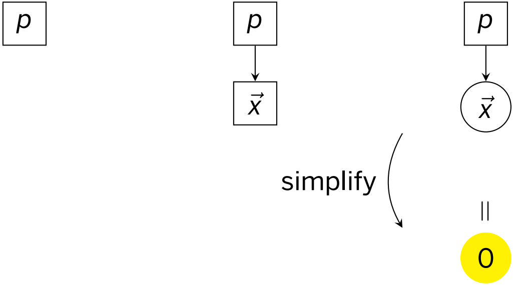
We assume that the position p of the particle is distributed normally with mean 3 and standard deviation 2.

We measure the particle 3 times, each time drawing independently from the normal distribution with mean p and standard deviation 1, and observe $-1.4, +1.0, -0.2$.

What is the distribution of the next measurement?

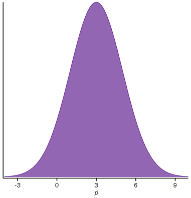
$$\begin{tikzcd}
\Pr(p) \arrow[r, bend left, "\text{bind}"] & \Pr(p, \vec{x}) \arrow[r, bend left, "\text{disintegrate}"] & \Pr(p \mid \vec{x}) \arrow[r, bend left, "\text{bind}"] & \Pr(p, y \mid \vec{x}) \arrow[r, bend left, "\text{integrate}"] & \Pr(y \mid \vec{x})
\end{tikzcd}$$

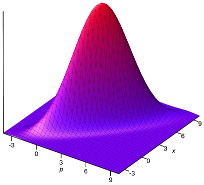


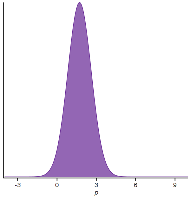


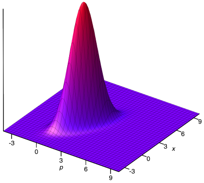
simplify

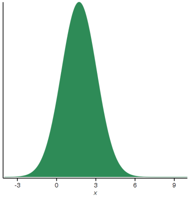
$$\begin{tikzcd}
\Pr(p) \arrow[r, bend left, "\text{bind}"] & \Pr(p, \vec{x}) \arrow[r, bend left, "\text{disintegrate}"] & \Pr(p \mid \vec{x}) \arrow[r, bend left, "\text{bind}"] & \Pr(p, y \mid \vec{x}) \arrow[r, bend left, "\text{integrate}"] & \Pr(y \mid \vec{x})
\end{tikzcd}$$

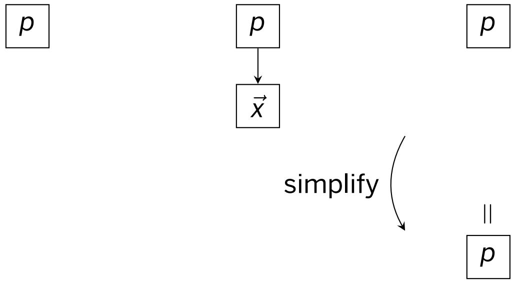












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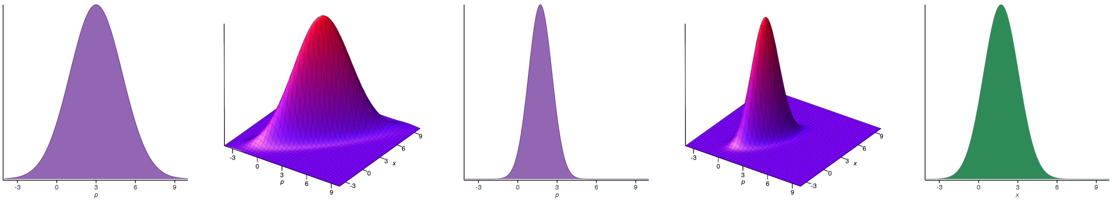
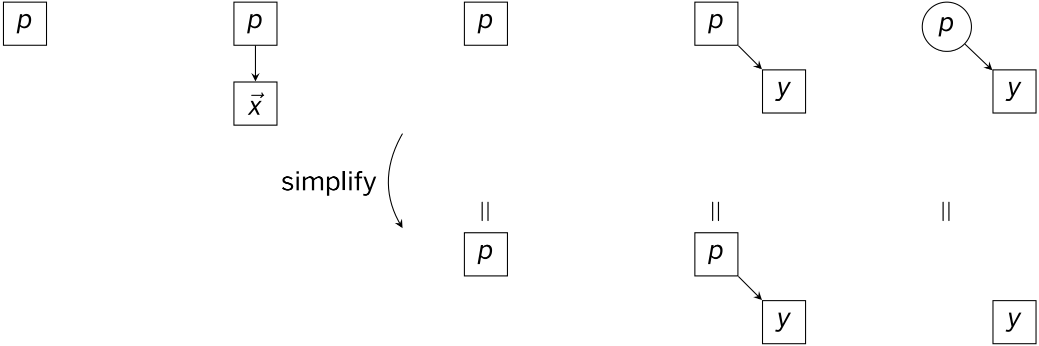
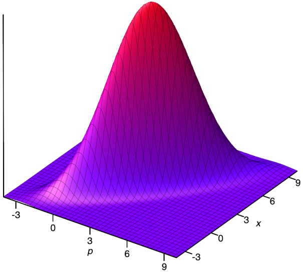
\Pr(p) \arrow[r, "\text{bind}"] & \Pr(p, \vec{x}) \arrow[r, "\text{disintegrate}"] & \Pr(p \mid \vec{x}) \arrow[r, "\text{bind}"] & \Pr(p, y \mid \vec{x}) \arrow[r, "\text{integrate}"] & \Pr(y \mid \vec{x})
\end{tikzcd}$$

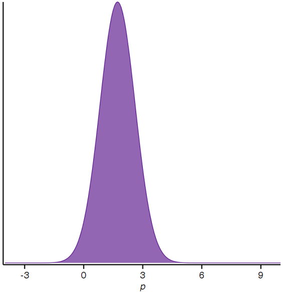


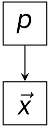

simplify

Disintegrating a joint measure

$$\Pr(p, \vec{x}) \xrightarrow{\text{disintegrate}} \Pr(p \mid \vec{x})$$





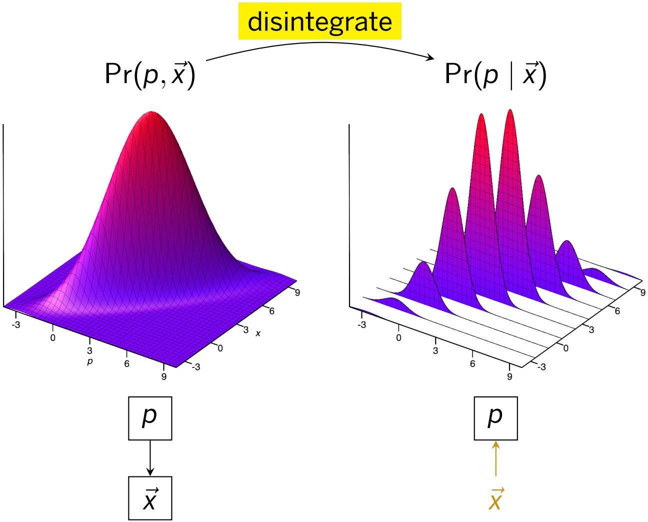


p

$\vec{x}$

p

Disintegrating a joint measure



disintegrate

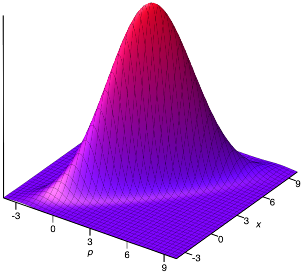
$\Pr(p, \vec{x}) \xrightarrow{\text{disintegrate}} \Pr(p \mid \vec{x})$

$\boxed{p} \rightarrow \boxed{\vec{x}}$

$\boxed{p} \leftarrow \vec{x}$

Disintegrating a joint measure

$$\Pr(p, \vec{x}) \xrightarrow{\text{disintegrate}} \Pr(p \mid \vec{x})$$



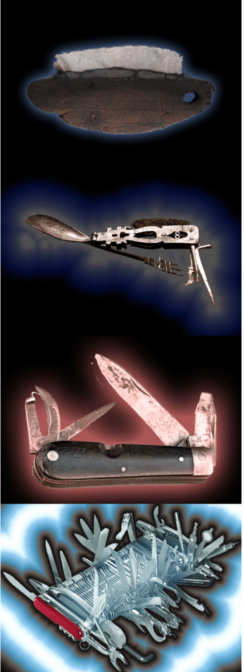
$$\begin{tikzcd}
\boxed{p} \arrow[d] & = & \boxed{p} \\
\boxed{\vec{x}} & & \boxed{\vec{x}} \arrow[u]
\end{tikzcd}$$

Program transformation derived from semantics.

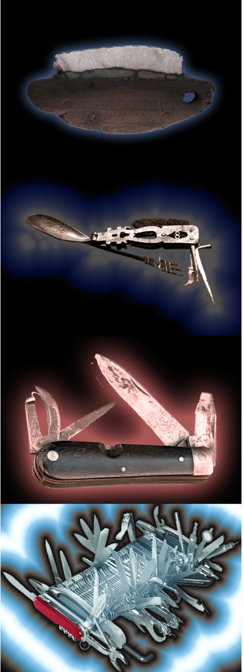
Tricky when $\vec{x}$ is not just drawn from a primitive distribution:

- total momentum
- loop over array
- clamped measurement
- coordinate-wise MCMC

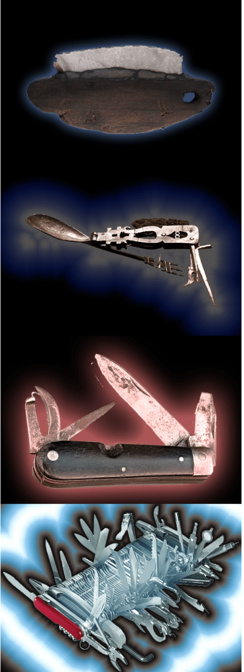
Addressed in recent work. (ICFP 2016, POPL 2017, ICFP 2017)



7	8	9	÷
4	5	6	×
1	2	3	−
0	.	=	+



7 *plate*	8 *lambda*	9 *apply*	÷ *disintegrate*
4 *pair*	5 *fst*	6 *snd*	× *bind*
1 *dirac*	2 *beta*	3 *normal*	− *gradient*
0 *mzero*	. *factor*	= *simplify*	+ *mplus*



Thanks!

Jacques Carette

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Praveen Narayanan

Norman Ramsey

Wren Romano

Sam Tobin-Hochstadt

Rajan Walia

Robert Zinkov

÷ ***disintegrate***

× ***bind***

− ***gradient***

0 ***mzero***

. ***factor***

= ***simplify***

+ ***mplus***