Reasoning about contexts in Henkin models

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Lambda Calculus and Formal Grammar
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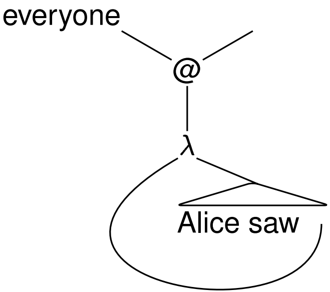
Contexts are syntactic functions

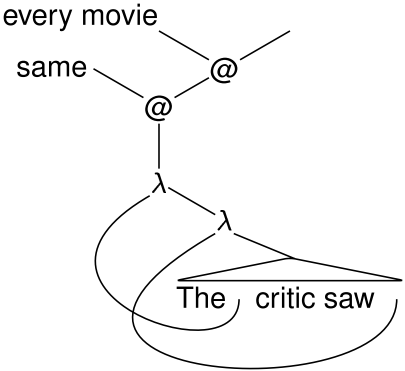
Alice saw [everyone].
everyone $\circ$ (λx. Alice saw x)



Alice saw [him].
him $\circ$ (λx. Alice saw x)

Contexts are syntactic functions

The [same] critic saw [every movie]. (Barker)
every movie $\circ$ (same $\circ$ ($\lambda x.\ \lambda y.$ The x critic saw y))



Americans [on average] have [2.3] children. (Kennedy & Stanley)
2.3 $\circ$ (on average $\circ$ ($\lambda x.\ \lambda y.$ Americans x have y children))

Contexts are syntactic functions

But not ‘exotic’ functions such as

$$\lambda x.\ \begin{array}{l}\text{if } x = \text{Bob}\\ \text{then Alice saw } x\\ \text{else } x \text{ slept}\end{array}$$

(cf. Henkin models for higher-order logic)

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This talk

From proof theory λ-terms and their β-equivalence

To model theory *relational* models of the λ-calculus

van Benthem 1999: Relating modal logic (categorial grammar)
and type theory (abstract categorial grammar)

NL proofs

Connectives: $\backslash$ $/$ Punctuation: $\bullet$

$$
\dfrac{\dfrac{\dfrac{}{\mathsf{DP} \vdash \mathsf{DP}} \quad \dfrac{}{\mathsf{S} \vdash \mathsf{S}}}{\mathsf{DP} \bullet \mathsf{DP}\backslash\mathsf{S} \vdash \mathsf{S}}\,\backslash L \qquad \dfrac{}{\mathsf{DP} \vdash \mathsf{DP}}}{\underbrace{\mathsf{DP}}_{\text{Alice}} \bullet (\underbrace{(\mathsf{DP}\backslash\mathsf{S})/\mathsf{DP}}_{\text{saw}} \bullet \underbrace{\mathsf{DP}}_{\text{Bob}}) \vdash \mathsf{S}}\,/L
$$

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$$

$$
\dfrac{}{X \vdash X} \qquad \dfrac{\Gamma \vdash X \quad \Sigma[Y] \vdash Z}{\Sigma[\Gamma \bullet X\backslash Y] \vdash Z}\,\backslash L \qquad \dfrac{X \bullet \Gamma \vdash Y}{\Gamma \vdash X\backslash Y}\,\backslash R
$$

$$
\dfrac{\Gamma \vdash X \quad \Sigma[X] \vdash Y}{\Sigma[\Gamma] \vdash Y} \qquad \dfrac{\Sigma[Y] \vdash Z \quad \Gamma \vdash X}{\Sigma[Y/X \bullet \Gamma] \vdash Z}\,/L \qquad \dfrac{\Gamma \bullet X \vdash Y}{\Gamma \vdash Y/X}\,/R
$$

NL models

A *frame* consists of

- a set of points $\mathcal{P}$ and
- a ternary accessibility relation $R_\bullet \subseteq \mathcal{P}^3$.

A *model* consists of a frame and a valuation $\Vdash$ that relates points p, q, r to the structures and formulas they satisfy.

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$$r \Vdash \mathsf{DP} \bullet ((\mathsf{DP}\backslash\mathsf{S})/\mathsf{DP} \bullet \mathsf{DP})$$

$$\iff \quad \exists p.\, \exists q.\, R_\bullet(p,q,r) \wedge (p \Vdash \mathsf{DP}) \wedge (q \Vdash (\mathsf{DP}\backslash\mathsf{S})/\mathsf{DP} \bullet \mathsf{DP})$$

$$r \Vdash (\mathsf{DP}\backslash\mathsf{S})/\mathsf{DP} \bullet \mathsf{DP}$$

$$\iff \quad \exists p.\, \exists q.\, R_\bullet(p,q,r) \wedge (p \Vdash (\mathsf{DP}\backslash\mathsf{S})/\mathsf{DP}) \wedge (q \Vdash \mathsf{DP})$$

$$p \Vdash (\mathsf{DP}\backslash\mathsf{S})/\mathsf{DP}$$

$$\iff \quad \forall q.\, \forall r.\, R_\bullet(p,q,r) \to (q \Vdash \mathsf{DP}) \to (r \Vdash \mathsf{DP}\backslash\mathsf{S})$$

$$q \Vdash \mathsf{DP}\backslash\mathsf{S}$$

$$\iff \quad \forall p.\, \forall r.\, R_\bullet(p,q,r) \to (p \Vdash \mathsf{DP}) \to (r \Vdash \mathsf{S})$$

NL soundness and completeness

$$\Gamma \vdash X \qquad \Longleftrightarrow \qquad \text{In any model, at any point } p\text{, if } p \Vdash \Gamma \text{ then } p \Vdash X.$$

The canonical completeness proof constructs a canonical model, in which points are structures.

NL$_\lambda$ proofs

Connectives: $\backslash$ $/$ $\backslash\!\!\backslash$ $/\!\!/$ Punctuation: $\bullet$ $\circ$ x λx (linear)

$$\dfrac{\dfrac{\dfrac{\dfrac{\vdots}{\mathsf{DP} \bullet ((\mathsf{DP}\backslash \mathsf{S})/\mathsf{DP} \bullet \mathsf{DP}) \vdash \mathsf{S}}}{\mathsf{DP} \circ \lambda x.(\mathsf{DP} \bullet ((\mathsf{DP}\backslash \mathsf{S})/\mathsf{DP} \bullet x)) \vdash \mathsf{S}}\,\beta}{\lambda x.(\mathsf{DP} \bullet ((\mathsf{DP}\backslash \mathsf{S})/\mathsf{DP} \bullet x)) \vdash \mathsf{DP}\backslash\!\!\backslash \mathsf{S}}\,\backslash\!\!\backslash R \qquad \dfrac{}{\mathsf{S} \vdash \mathsf{S}}}{\dfrac{\mathsf{S}/\!\!/(\mathsf{DP}\backslash\!\!\backslash \mathsf{S}) \circ \lambda x.(\mathsf{DP} \bullet ((\mathsf{DP}\backslash \mathsf{S})/\mathsf{DP} \bullet x)) \vdash \mathsf{S}}{\underbrace{\mathsf{DP}}_{\text{Alice}} \bullet (\underbrace{(\mathsf{DP}\backslash \mathsf{S})/\mathsf{DP}}_{\text{saw}} \bullet \underbrace{\mathsf{S}/\!\!/(\mathsf{DP}\backslash\!\!\backslash \mathsf{S})}_{\text{everyone}}) \vdash \mathsf{S}}\,\beta}\,/\!\!/ L$$

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$$\dfrac{\Sigma[\Gamma[\Delta]] \vdash X}{\Sigma[\Delta \circ \lambda x.\, \Gamma[x]] \vdash X}\beta$$

For binding, [same], and [on average], the context $\Gamma[\,]$ may contain λ.

NL_λ models

A *frame* consists of

- a set of points $\mathcal{P}$,
- a ternary accessibility relation $R_\bullet \subseteq \mathcal{P}^3$, and
- a ternary accessibility relation $R_\circ \subseteq \mathcal{P}^3$ (not a function),

such that there are 'enough functions' (more on that later).

NL_λ models: satisfaction

How to define satisfaction?
Points in structures are convenient (hybridization).

$$p \Vdash q$$

$$\iff \quad p = q$$

$$q \Vdash \lambda x.(\mathsf{DP} \bullet ((\mathsf{DP}\backslash\mathsf{S})/\mathsf{DP} \bullet x))$$

$$\overset{?}{\iff} \quad \forall p.\, \forall r.\, R_\circ(p, q, r) \leftrightarrow (r \Vdash \mathsf{DP} \bullet ((\mathsf{DP}\backslash\mathsf{S})/\mathsf{DP} \bullet p))$$

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But what if no point satisfies DP or no point satisfies $(\mathrm{DP}\backslash\mathrm{S})/\mathrm{DP}$?

$$\dfrac{\dfrac{\mathrm{DP} \circ \lambda y.\, \lambda x.(y \bullet ((\mathrm{DP}\backslash \mathrm{S})/\mathrm{DP} \bullet x)) \vdash X}{\lambda x.(\mathrm{DP} \bullet ((\mathrm{DP}\backslash \mathrm{S})/\mathrm{DP} \bullet x)) \vdash X}\,\beta}{(\mathrm{DP}\backslash \mathrm{S})/\mathrm{DP} \circ \lambda z.\, \lambda x.(\mathrm{DP} \bullet (z \bullet x)) \vdash X}\,\beta$$

NL_λ models: satisfaction

The definition of satisfaction for $\lambda x.\,\Gamma[x]$ quantifies over the maximal substructures of $\Gamma[x]$ that do not contain x.

$$p \Vdash q \quad\Longleftrightarrow\quad p = q$$

$$\begin{aligned} & q \Vdash \lambda x.\big(\mathsf{DP} \bullet ((\mathsf{DP}\backslash\mathsf{S})/\mathsf{DP} \bullet x)\big) \\ \Longleftrightarrow \quad & \exists s.\,\exists t.\,(s \Vdash \mathsf{DP}) \wedge (t \Vdash (\mathsf{DP}\backslash\mathsf{S})/\mathsf{DP}) \\ & \qquad \wedge \forall p.\,\forall r.\,R_\circ(p,q,r) \leftrightarrow (r \Vdash s \bullet (t \bullet p)) \end{aligned}$$

Each λ-abstraction shape is like a jumbo product connective.

$$\dfrac{\dfrac{\mathsf{DP} \circ \lambda y.\,\lambda x.\big(y \bullet ((\mathsf{DP}\backslash\mathsf{S})/\mathsf{DP} \bullet x)\big) \vdash X}{\lambda x.\big(\mathsf{DP} \bullet ((\mathsf{DP}\backslash\mathsf{S})/\mathsf{DP} \bullet x)\big) \vdash X}\;\beta}{(\mathsf{DP}\backslash\mathsf{S})/\mathsf{DP} \circ \lambda z.\,\lambda x.\big(\mathsf{DP} \bullet (z \bullet x)\big) \vdash X}\;\beta$$

NL$_\lambda$ models: the environment model condition

We require of the frame that there be 'enough functions':

- There must be some point q such that $q \Vdash \lambda x.\, x$.
- For any points s, t, there must be some point q such that $q \Vdash \lambda x.\, s \bullet (t \bullet x)$.
- And so on, for each λ-abstraction shape.

Or in computational terms: we can always build a closure.

NL$_\lambda$ soundness and completeness

$\Gamma \vdash X \quad \iff \quad$ In any model, at any point p, if $p \Vdash \Gamma$ then $p \Vdash X$.

The canonical completeness proof constructs a canonical model, in which points are β-equivalence classes of structures.

NL_λ conservativity over NL

An NL sequent that is provable in NL_λ is already provable in NL.

Extend any NL model to an NL_λ model whose points are β-equivalence classes of structures whose maximal substructures that do not contain variables are the old points.

What keeps the old points separate in the new model is the confluence of the λ-calculus!

Domain theory for syntax?

Summary

Relational models of the λ-calculus

- are natural to define;
- capture the meanings of contexts as syntactic functions;
- should perhaps be equipped with kind structure.