A Triangular Adventure

Douglas R. Hofstadter
Center for Research on Concepts & Cognition
Indiana University, Bloomington
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It all starts with an elegant construction and surprising discovery made by Karleen Davis and Sam Johnson, students in my "Cat:Dog" ("Circles and Triangles: Diamonds of Geometry") class this semester.

Karleen and Sam were just playing around, looking for interesting things to do with a triangle. One thing they tried was this. Starting with a random triangle ABC, they took H, its orthocenter, and O, its circumcenter, and drew the three cevians through H (a cevian is any line passing through a vertex) and the three cevians through O. Each H-cevian meets two of the O-cevians (and vice versa), making six intersection points in all, which can be denoted as follows:

> Ab = the intersection of cevian OA with cevian HB;
> Ba = the intersection of cevian OB with cevian HA; etc.

Karleen and Sam divided these six points up into two natural trios, constituting the vertices of two natural triangles (shown in Figure 1):

> triangle 1 = Ab–Bc–Ca;
> triangle 2 = Ba–Cb–Ac.

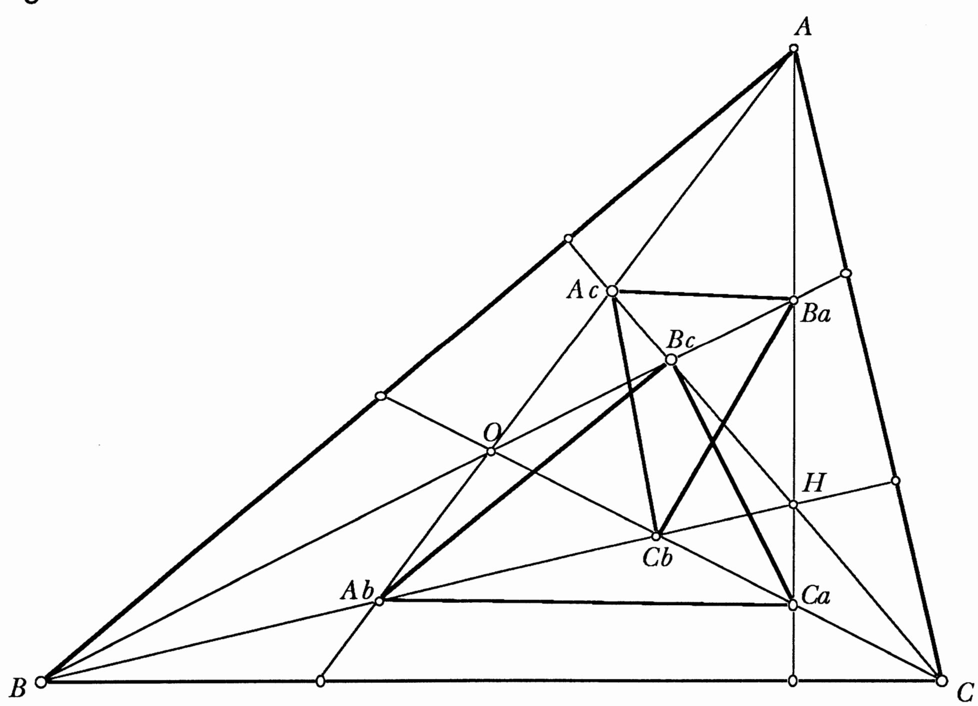

Figure 1.

In a sense, one of these "junior triangles" involves making a tour of triangle ABC in a clockwise direction, and the other makes the tour in counterclockwise fashion.

Karleen and Sam noticed that no matter what shape ABC is, both new triangles are precisely <u>similar</u> to it (the angles in Figure 2 show it is true in this case).

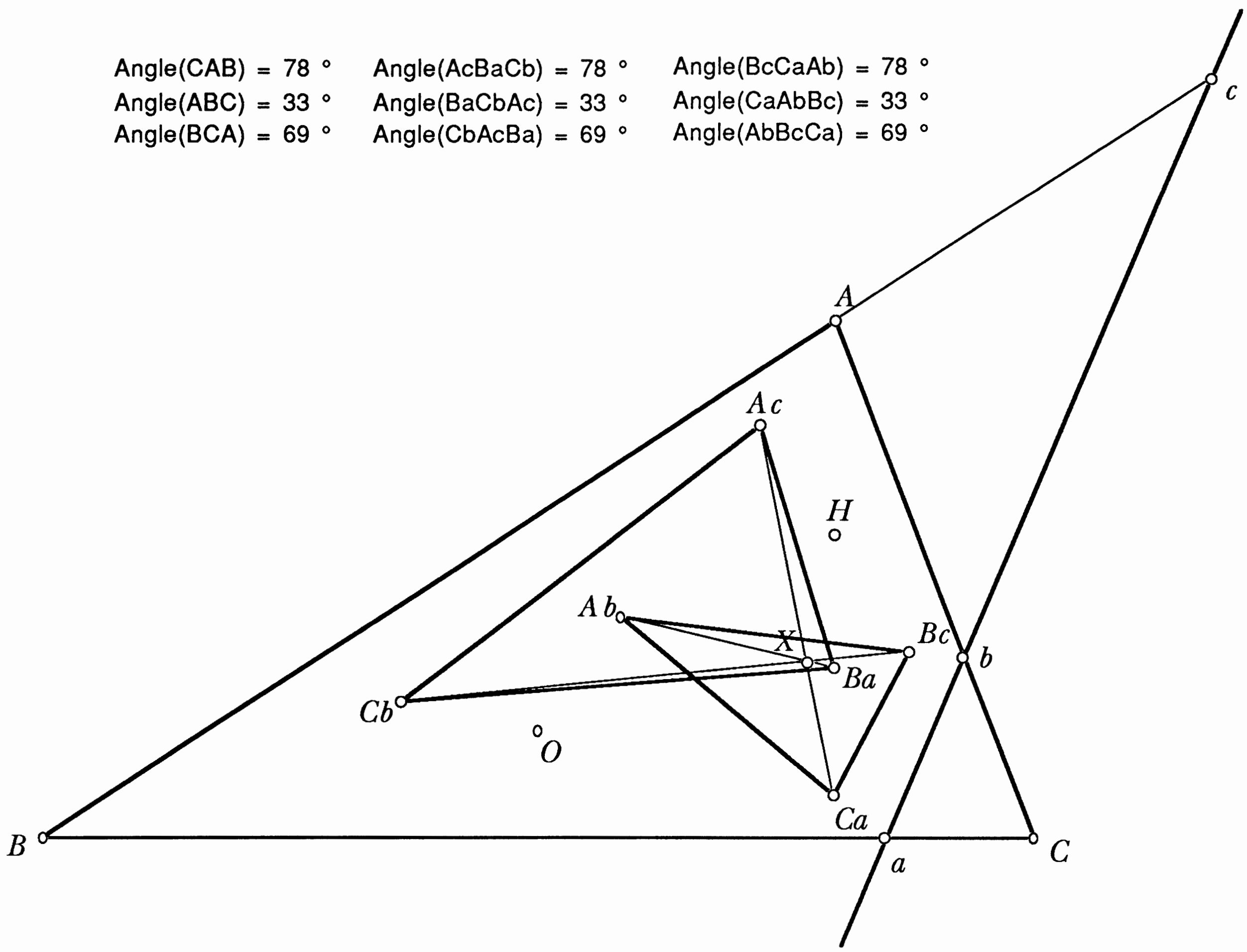

Figure 2.

Moreover, they noticed that when "opposite" vertices (e.g., Ab and Ba) were joined by lines, the three lines thus formed all met in a single point, here denoted "X". Triangles with this latter property are said to be perspective, and the point where the three lines meet is called the center of perspectivity.

This is the stage where Karleen and Sam showed me their discovery. They hadn't proven anything yet, and were wondering where to go with it. One obvious open question was, what role does this point X play in the original triangle? Is it a well-known point, or a known but obscure point, or is it a completely new point? Just eyeballing it, I couldn't make any guesses, but it certainly felt like this X-point should be a significant point.

Leaving aside the question of the identity of X, the other aspects of their discovery (similarity and perspectivity) struck me as quite wonderful, but I felt that it would be truly astonishing if these or similar properties held only in case one began with O and H. It would be too much of a coincidence if Karleen and Sam had just by chance chosen the only two points that gave anything interesting. Much more likely, it seemed to me, was that their finding would generalize, and therefore that the first thing they should try was to try other starting-points. It occurred to me that, given that O and H are endpoints of the famous Euler segment, the endpoints of the tightly analogous "Nagel segment" might be a natural next place to look. (These are I, the incenter, and N, the Nagel point.) But I also thought that they should try out their twin-triangles construction starting with a variety of pairs of famous centers — the Gergonne point, the Fermat point, and so on. That was one suggestion I made.

Another was that Desargues' theorem might be relevant here. This theorem says that triangles perspective from a point are also perspective from a line. More specifically, it says that if two triangles are perspective from a point, then when corresponding sides of the triangles are intersected pairwise, the three points thus formed will be collinear. I mentioned this to them, and then we concluded our session.

That evening, I was very curious to follow this out a bit further. I decided to look first of all at my idea involving Desargues' theorem, sticking temporarily with O and H as starting-points. So I turned on Geometer's Sketchpad and made the twin-triangles construction, confirming Karleen and Sam's results. It was certainly very pretty to watch the "dance" of the two triangles inside triangle ABC. But soon I turned to exploration of the Desargues idea. I wanted to see the axis of perspective on my screen.

Upon intersecting sides Ca–Ab and Ba–Ac, I was quite surprised to find that their meeting-point, which I called "a", seemed to lie on side BC of the original triangle. Of course, I then found that point b, the intersection of sides Ab–Bc and Cb–Ba, lay on side CA, and point c, the intersection of sides Bc–Ca and Ac–Cb, lay on side AB. To repeat, what was interesting here was not that these three points were collinear — I knew that in advance, thanks to Desargues' theorem; however, what I didn't expect at all was that they would lie on the sides of the original triangle ABC.

At this point, I called up Clark Kimberling to tell him of this unusual cluster of discoveries, and like me, he too intuitively sensed that other pairs of starting-points had to be tried out. He observed that one thing that links O and H is that they are isogonal conjugate points (the details don't matter), and suggested that that might be the key — in other words, perhaps interesting properties would arise if you started with any pair of isogonal conjugates. This seemed like a reasonable hunch.

So before trying out my own hunch of using I and N, I took Clark's suggestion and took a random point P and constructed its isogonal conjugate P', and then carried through the whole twin-triangles construction with P and P' as starting-points. What I found was that some parts of Karleen and Sam's "theorem" indeed did carry over, but that some parts failed to. On the one hand, the two new triangles failed to be similar to the original one, but on the other hand, they were still perspective from a point and thus from a line, and the three points determining that line — the axis of perspective — were still on the sides of the original triangle. This was very interesting news, and made me think that Clark's intuition was very keen. But it then occurred to me that perhaps one could go further than this. Maybe just any two points would work. Maybe it was a red herring to think that the starting-points needed in any way at all to be special or related to each other.

So I made the ultimate leap of faith, if that's the right term, and started afresh with two random points, P and Q, making the same twin-triangles construction. Somewhat to my delight and somewhat to my disappointment, I found that everything worked just as well for P and Q as for P and its isogonal conjugate. So Clark's intuition hadn't been so keen, after all. Like me, he had at first been tricked by the fact that O and H were the original starting-points into thinking that perhaps centers were somehow needed to make things work out. It's true that something had been lost when O and H were let go of — namely, the similarity of the twin triangles to ABC and perforce to each other — but surely, this was a deeper invariance. But where was it coming from?

A key insight came at this point through a mistake on my part. In trying to make the diagram on my screen simpler and cleaner, I deleted the lines defining point X, but then I decided it wasn't that helpful, so I wanted to put them back. In trying to do so, I was a little careless, and didn't look at the labels on the points of the twin triangles. Instead of joining Ab to Ba, I joined it to Ac. Similarly, I joined Ba to Bc, and Ca to Cb. I found that these three lines intersected in a point, but it surely wasn't X (see Figure 3).

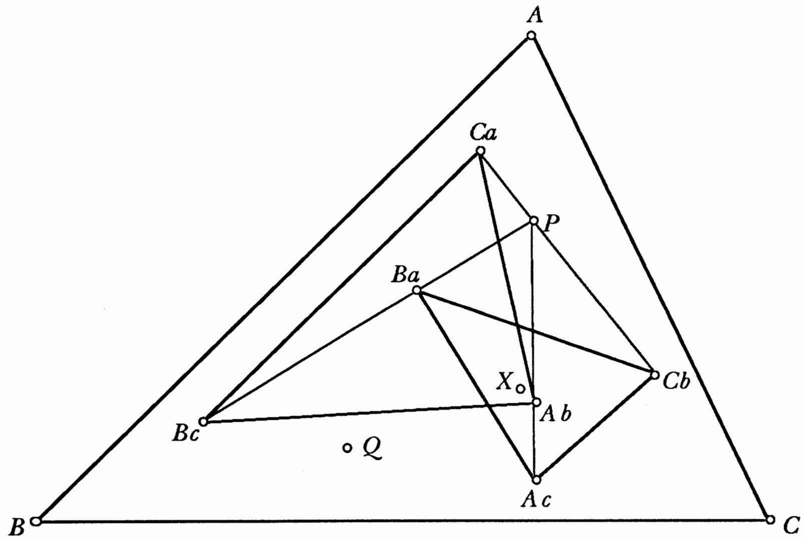

Figure 3.

In fact, it was P itself! Why was this? I immediately realized that these three lines were nothing but the cevians going through point P. Of course these lines all met in P! It was trivial. And similarly, the lines joining Ac to Bc, Ca to Ba, and Ab to Cb all meet in Q, because they are none other than the three remaining cevians. I felt quite dumb for not having anticipated this triviality — and yet all of a sudden the obvious fact hit me: triangles 1 and 2 are not just perspective from X, they are also perspective from P and from Q! This is the very essence of the original construction!

It felt as if I should have seen this instantly, but I hadn't. It hadn't occurred to me to think of things that way, even though it was staring me in the face. But now I remembered a theorem I had read a few days earlier, and which had sounded elegant at the time but unconnected to anything I had experience with. That theorem said: "If two triangles are doubly perspective, they are triply perspective." More concretely, if two triangles IJK and LMN are such that lines IL, JM, and KN are concurrent and also lines IM, JN, and KL are concurrent, then lines IN, JL, and KM will also be concurrent.

All of a sudden, this theorem was taking on a concrete meaning. In particular, I saw that the existence of Karleen and Sam's X point followed immediately from this theorem. Namely, the construction itself establishes the fact that triangles 1 and 2 are doubly perspective, once from P and once from Q; it follows that they are triply perspective, so the remaining lines all have to come together at some point, QED.

Once I saw this, I realized that a lot more was going on in the original diagram than had hit my eye. In particular, I saw that one could interpret the cevians through P as constituting a perspectivity between triangle ABC and triangle 1, in which P once again plays the role of center of perspective. Moreover, those same three cevians simultaneously can be seen as constituting a perspectivity between triangle ABC and triangle 2, with P again being the center of perspective. It was getting a little boring!

Of course, exactly the same things could be said about the cevians through Q — they establish the fact that ABC and triangle 1 are perspective from Q, as well as the fact that ABC and triangle 2 are perspective from Q.

When I put this all together in my mind, I realized that I had two more cases of doubly perspective triangles on my hands — triangle 1 was doubly perspective with ABC from both P and Q, and likewise for triangle 2. So this meant there had to be two further centers of perspective, one linking ABC with triangle 1 (I called it "Pq"), the other linking ABC with triangle 2 (I called it "Qp"). When I constructed these new centers of perspective, I was in for another surprise: X was collinear with them. All this can be seen in Figure 4.

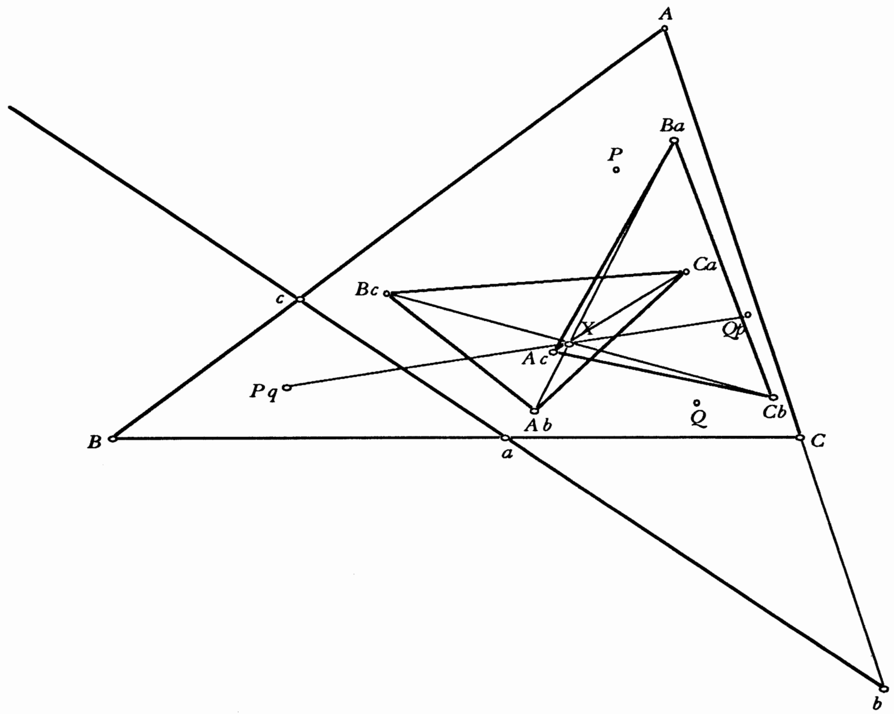

Figure 4.

Here is a little table summarizing most of what I had found so far. (In the table, "triangle 0" means triangle ABC.)

Triangles	Centers of perspective
1 and 2	P, Q, X
0 and 1	P, Q, Pq
0 and 2	P, Q, Qp

Two main findings are not represented in the table. One is the collinearity of the centers of perspective Pq, X, and Qp; the other is the surprise involving the locations of a, b, and c, the points that define the axis of perspective linking triangles 1 and 2.

This brings us back to the whole idea of axes of perspective. After all, Desargues tells us that for each center of perspective, there is an axis of perspective. This means there are eight further significant lines that belong in the picture!

We begin by looking at the axes corresponding to centers Pq and Qp. If we accept the experimental observation (certain to be true, but unproven so far) that points a, b, and c do lie on the sides of triangle 0, this means they are perforce also the sites where sides of triangles 0 and 1 meet (and likewise for triangles 0 and 2). As a consequence, line abc is not just the axis corresponding to center of perspective X — it is also the axis corresponding to center of perspective Pq <u>and</u> the axis corresponding to center of perspective Qp! Thus line abc plays a triply degenerate role as an axis of perspective in this figure.

The remaining six axes of perspective — which I will call p01, p02, p12, and q01, q02, q12 — turn out all to be distinct lines, but constructing them reveals that they too have a surprising property — namely, they are all concurrent in a single point, namely point M. This can be seen in Figure 5.

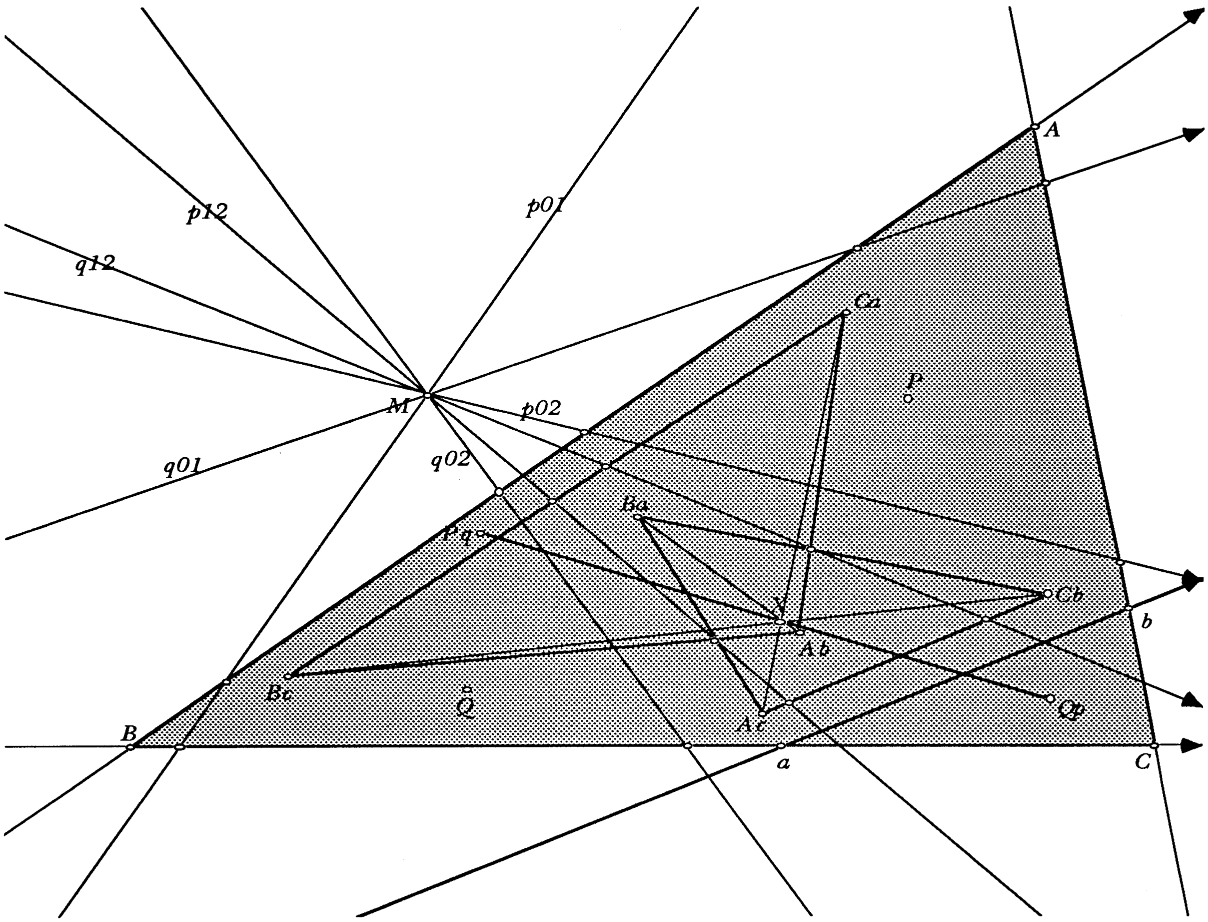

Figure 5.

The table of axes of perspective, corresponding to the earlier table of centers of perspective, is as follows:

Triangles	Axes of perspective
1 and 2	p12, q12, abc
0 and 1	p01, q01, abc
0 and 2	p02, q02, abc

All in all, a good number of surprises had been discovered. First was the similarity of triangles 1 and 2 to triangle ABC (and perforce to each other) when H and O were used as starting-points. Second was the fact that triangles 1 and 2 were perspective.

The next surprise was the fact that the perspectivity of triangles 1 and 2 was preserved no matter what points were used to begin with. Then came the surprise that the points determining the axis of perspective lay on the sides of triangle ABC. Then the surprise that there were in fact nine centers of perspective in all, of which six collapsed down into just two triple centers (points P and Q), and the remaining three of which were collinear (Pq, X, Qp). Then came two more surprises: first, that line abc was a triple axis of perspective, and second, that the remaining six axes of perspective were all concurrent.

Are all these surprises essentially one surprise? That is, is there just one little theorem to prove from which they all then drop out as simple consequences? I suspect so.

It is of interest to point out that whereas Karleen and Sam's original result involving special points O and H belongs unambiguously to Euclidean plane geometry, all my findings (involving arbitrary points) belong to projective geometry, because they make no reference to lengths or to angles; all they involve is matters of incidence: whether a point lies on a given line, or whether a line goes through a given point. Now one of the most salient features of projective geometry is duality: the complete interchangeability of points with lines. Every theorem of projective geometry has a dual counterpart — a theorem structurally identical to the original, but with a reversal of terms, so that "point" and "line" are interchanged, as are "lie on" and "pass through", and so on. In some sense, such a pair of dual theorems actually constitutes just a single theorem of projective geometry, no matter how different they may seem on the surface.

To illustrate this, consider the dual of my results (see Figure 6, below). One begins as before with a triangle. (This is because the concept of "triangle" is its own dual — it's simply that the roles of vertices and sides are interchanged.) However, instead of drawing two arbitrary points P and Q, one draws two arbitrary lines, p and q. Each of these lines determines not three cevians (which are lines through vertices), but three menelaians (which are points along sides). Specifically, the menelaians determined by any line are the three points where that line pierces the (extended) sides of the triangle. Thus the concept of "menelaian" is the dual of the concept of "cevian".

The next step is to construct two twin triangles, and to do so via the making of new lines (which will serve as the triangles' sides) rather than new points (which in our earlier construction served as the triangles' vertices). But this is easy: pairs of menelaians on different sides of triangle 0 will determine new lines. Thus, for instance, pa (the p-menelaian on side BC) determines a new line with qb (the q-menelaian on side CA). Altogether, taking menelaians in such pairs yields six new lines in all, which can be denoted as follows:

aB = the line joining menelaian pa with menelaian qb;
bA = the line joining menelaian pb with menelaian qa; etc.

In perfect analogy to what we did before, we now divide these six lines up into two natural trios, constituting the sides of two natural triangles:

triangle 1 = aB–bC–cA;
triangle 2 = bA–cB–cA.

It is now by construction (rather than by Desargues' theorem) that triangles 1 and 2 are perspective from line p, as well as from line q. And of course two triangles that are doubly perspective are triply perspective, meaning that there must be a third line, line x, from which triangles 1 and 2 are perspective. Given that this new line exists, Desargues' theorem (or more properly, its dual) tells us that there is a special point from which triangles 1 and 2 are perspective. To find this point, we of course connect up corresponding vertices pairwise, thus making three lines A, B, and C, which are concurrent in a point, called point ABC.

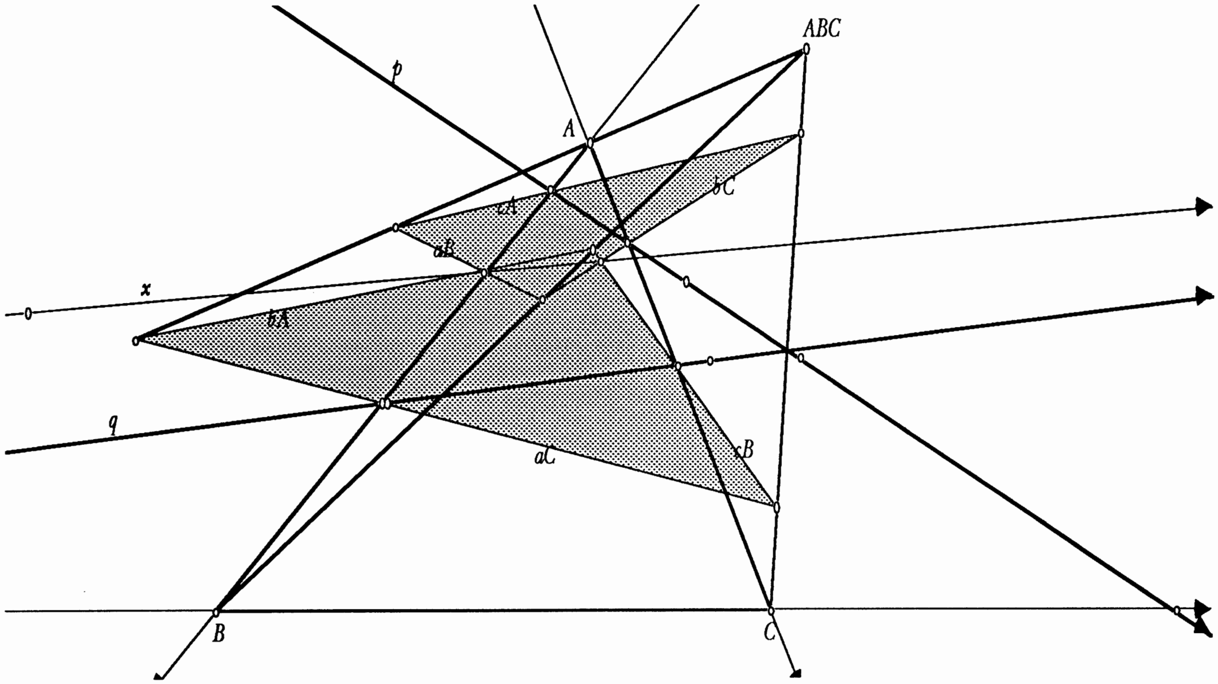

Figure 6

What is the analogue to the surprising finding, in the original case, that points a, b, and c lay on the sides of triangle ABC (i.e., were menelaians)? Of course it must be that each of our new lines is in fact a cevian of the original triangle ABC. Needless to say, just as line abc was a triply degenerate axis of perspective, so point ABC is a triply degenerate center of perspective.

We could go on, eventually winding up with a set of nine centers of perspective for the three triangles, three of which are of course located at point ABC, and the other six of which will be collinear, on a line called "m". All of this would be a little too messy to construct, but it most assuredly is the case, since it is nothing but the dual to the previously found result.

Thus we have wandered truly a long ways from Karleen and Sam's original discovery. Let us close by returning precisely to their finding. What made them interested in what they saw before them on the screen was the fact that the two twin triangles were both similar to the original triangle. Had that not been the case, they would have felt no compulsion to pursue the properties of their set-up any further, and the book would have been closed before it even was opened at all! So all of these findings are in fact dependent on the coincidence that Karleen and Sam began by making cevians belonging to the special points O and H. Perhaps this suggests that it is worthwhile pursuing that very special case more than we did. And so we are brought back to the question, "What is the meaning of the the X-point in the original construction?" It remains to be seen.