From Euler to Ulam: Discovery and Dissection of a Geometric Gem

Douglas R. Hofstadter
Center for Research on Concepts & Cognition
Indiana University • 510 North Fess Street
Bloomington, Indiana 47408

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Chapter O
Bewitched... by Circles, Triangles, and a Most Obscure Analogy

Although many, perhaps even most, mathematics students and other lovers of mathematics sooner or later come across the famous *Euler line,* somehow I never did do so during my student days, either as a math major at Stanford in the early sixties or as a math graduate student at Berkeley for two years in the late sixties (after which I dropped out and switched to physics). Geometry was not by any means a high priority in the math curriculum at either institution. It passed me by almost entirely.

Many, many years later, however, and quite on my own, I finally did become infatuated — nay, *bewitched* — by geometry. Just plane old Euclidean geometry, I mean. It all came from an attempt to prove a simple fact about circles that I vaguely remembered from a course on complex variables that I took some 30 years ago. From there on, the fascination just snowballed. I was caught completely off guard. Never would I have predicted that Doug Hofstadter, lover of number theory and logic, would one day go off on a wild Euclidean-geometry jag! But things are unpredictable, and that's what makes life interesting.

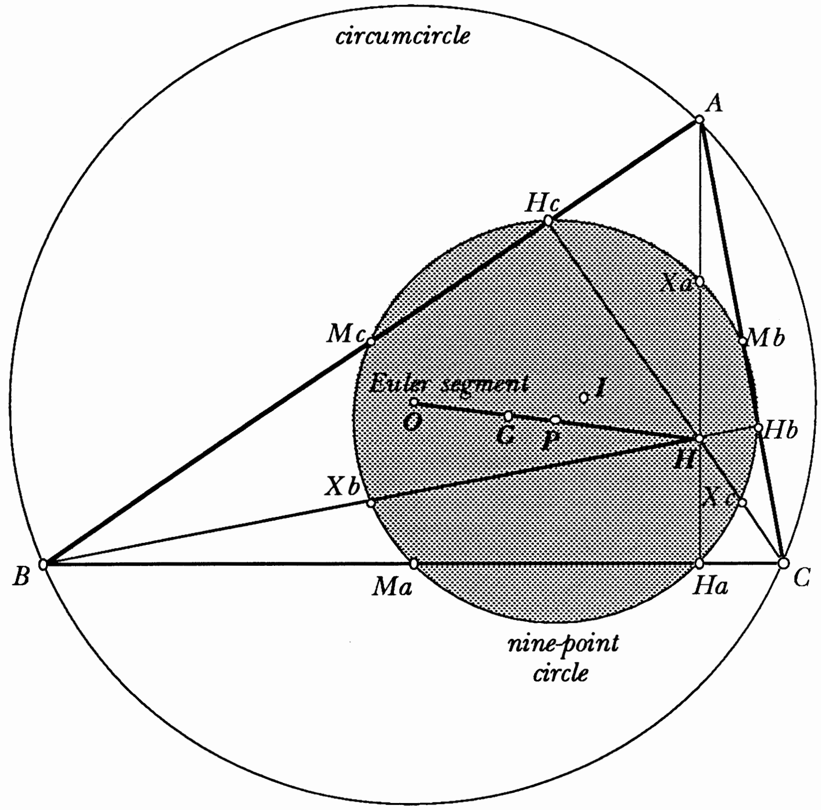
I especially came to love triangles, circles, and their unexpectedly profound interrelations. I had never appreciated how intimately connected these two concepts are. Associated with any triangle are a plentitude of natural circles, and conversely, so many beautiful properties of circles cannot be defined except in terms of triangles.

Not surprisingly, some of the most important points of a triangle are the *centers of various natural circles* associated with it. Three such circles are the *circumcircle,* which is the smallest circle that the triangle will fit inside, and which thus passes through all the triangle's vertices; the *incircle,* which is the largest circle that will fit inside the triangle, and which is thus tangent with all three of its sides; and the *nine-point circle,* a circle that somehow manages to pass through the midpoints of all three sides, the feet of all three altitudes, and three further notable points. The centers of these circles are the *circumcenter O,* the *incenter I,* and the *nine-point center P.*

Two other famous special points are the *centroid G* (also known as the "barycenter"), which is the center of gravity of the triangle (meaning that the triangle would balance perfectly if were supported by a pin located precisely at its centroid), and the *orthocenter H,* which is where all three altitudes cross (an interesting fact in itself, that they all cross in a single point!).

The Euler line or, as it more properly ought to be called, the Euler *segment* (which term I will use henceforth) connects four out of these five most special of special points. To be specific, the circumcenter, the centroid, the nine-point center, and the orthocenter all lie on a single line. (Poor little neglected incenter!) Shown below is a typical triangle *ABC* with its circumcircle, its nine-point circle, its three midpoints, its three altitudes, and its Euler segment *OGPH.*

The Euler segment runs from *O* to *H,* passing through both *G* and *P* en route, and always in that order. But it does more than that — it also possesses a *fixed set of length ratios.* Namely, *OG* is one-third of the length of the whole segment, and *OP* is one-half of the whole. (Incidentally, for fans of projective geometry, this means that O and P are harmonic conjugates with respect to G and H.)



The Euler line for triangle ABC — actually, the Euler segment — runs from ABC's circumcenter O to ABC's orthocenter H (where the altitudes crisscross). Precisely one-third of the way from O to H, the line passes through ABC's centroid G; at the exact halfway point, it passes through P, center of the so-called nine-point circle. The nine points through which that circle passes are: Ma, Mb, Mc (ABC's median points); Ha, Hb, Hc (feet of ABC's altitudes); and Xa, Xb, Xc (midpoints of the segments connecting H with each of the vertices). Last but not least, note the poor forgotten incenter I, somehow left out of the party.

When I learned that this holds for every triangle, I was riveted by the almost mystical-seeming interconnections thus revealed. I intuitively felt that this segment must represent something very deep about the original triangle *ABC* — something like its "essence", if there were such a thing. Could it be that the Euler segment is a kind of "key" containing, in some elegantly coded form, information about *ABC*'s many other special points, such as its incenter, its Fermat point (the point with the minimal sum of distances to the vertices), its Brocard center, its Morley center (and on and on the list goes), as well as about their hidden interrelationships?

Although I loved the Euler segment, I was deeply puzzled as to why the incenter I had been excluded from it, and felt that the incenter surely had to have its own special way of relating to these four points, or else, perhaps, its own coterie of special friends (although which ones they might be, I had no hunch about). Unresolved questions like this can lure one on, ever more deeply, into the study of special points and their unexpected hidden patterns. In any case, I was certainly hooked by these questions.

As I grew more involved with triangles, I started to see a metaphorical connection between my love for their special points and a mathematical love I had had from childhood: the love for *special points on the number line,* of which the quintessential examples are of course π and e. Among my other favorites were $\sqrt{2}$, the golden ratio ϕ, and Euler's constant γ. Perhaps the most exciting aspect of math for me was learning of equations that showed secret links among such numbers, such as Euler's equation:

$$e^{i\pi} = -1$$

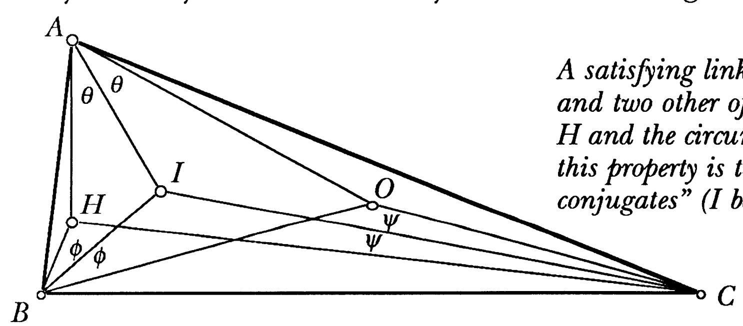
When I first saw this, perhaps at age 12 or so, it seemed truly magical, almost otherworldly. One can even draw an analogy between this equation, which relates four important numbers in a most astonishing way, and the Euler segment, which relates four important triangular points in a most astonishing way.

Of course, one difference between special points in a triangle and special points on the number line is that the former are *not constants* — they vary as the triangle itself varies. But this is somewhat misleading. What really counts about a given special point is not the physical point itself, but its *properties,* and these properties do *not* vary from one triangle to another. A typical theorem about special points tells you that Description 1 and Description 2, which on the surface seem to have nothing to do with each other, point at the very same spot on the plane. Seeing where that spot is, while nice, is not what matters — it's the abstract connection between descriptions that really counts. And that's a triangle-invariant property. Similarly, a typical theorem about special constants tells you that Formula 1 and Formula 2, which on the surface seem to have nothing to do with each other, point at the very same spot on the number line. One does not care so much about the number's *magnitude* (few mathematicians know more than a couple of decimals of e or π, or care what they are) as the fact that that spot marks the nexus of two very different conceptualizations.

Some of Indian mathematician Srinivasa Ramanujan's most amazing discoveries are amazing precisely because they have this quality of telling you that two unbelievably unrelated-looking expressions turn out to have exactly the same value. Nobody ever calculates or writes out the value, though. To concentrate on that would really be to miss the point. (Incidentally, Ramanujan himself was quite a mystic, and he often had no proofs for his results, claiming instead that they had been given to him by the goddess Namagiri in his dreams.)

Perhaps thoughts like these seem naïvely extra-mathematical, and irrelevant to the pursuit of mathematical truth. But I am convinced that the contrary is the case. That is, I believe that precisely these kinds of undeniably emotional, somewhat irrational reactions to things are the true motivators of the quest for mathematical truth, which is based, after all, on a sense of beauty, something that one certainly cannot put one's finger on with any kind of cool objectivity.

In any case, one day I made a little discovery on my own, which can be stated in the following picturesque way: If you are standing at any vertex and you swing your gaze from the circumcenter to the orthocenter, then when your head has rotated exactly halfway between them, you will be staring straight at the incenter.



A satisfying link between triangle ABC's incenter I and two other of its special points — the orthocenter H and the circumcenter O. One way of characterizing this property is to say that O and H are "isogonic conjugates" (I being its own isogonic conjugate).

More formally, the bisector of the angle formed by the two lines joining a given vertex V with the circumcenter and with the orthocenter points straight at the incenter. It wasn't too hard to prove this, luckily.

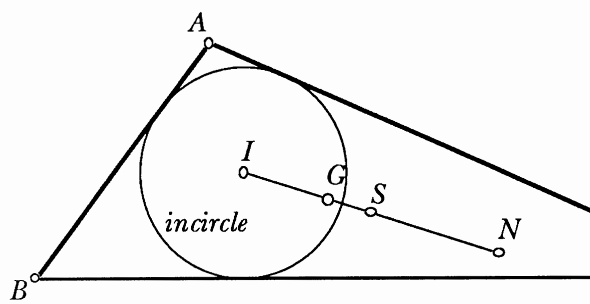
This discovery, which I knew must be as old as the hills, was a relief to me, since it somehow put the incenter back in the same league as the points I felt it deserved to be playing with. Even so, it didn't seem to play nearly as "central" a role as I felt it merited, and I was still a bit disturbed by this imbalance, almost an injustice.

Seeking to quench my avid thirst for geometrical insight, I went to a couple of superb technical bookstores with row after row after row of math books — but even there, all I found on geometry was a handful of rather thin volumes. These days, books devoted to this kind of topic — even books that devote a *chapter* to this kind of topic

— are rare as hen's teeth. It seems they went out of vogue at about the same time as the bee's knees and the cat's pyjamas (if not considerably earlier). In any case, I bought all the relevant books I could find, of which my favorite was *Geometry Revisited,* by H. S. M. Coxeter and Samuel Greitzer. None of these books on its own was anywhere close to definitive, but I nonetheless drank from them all with great gusto. And I must say, absorbing bits from all of them gave me quite a broad feeling for the subject.

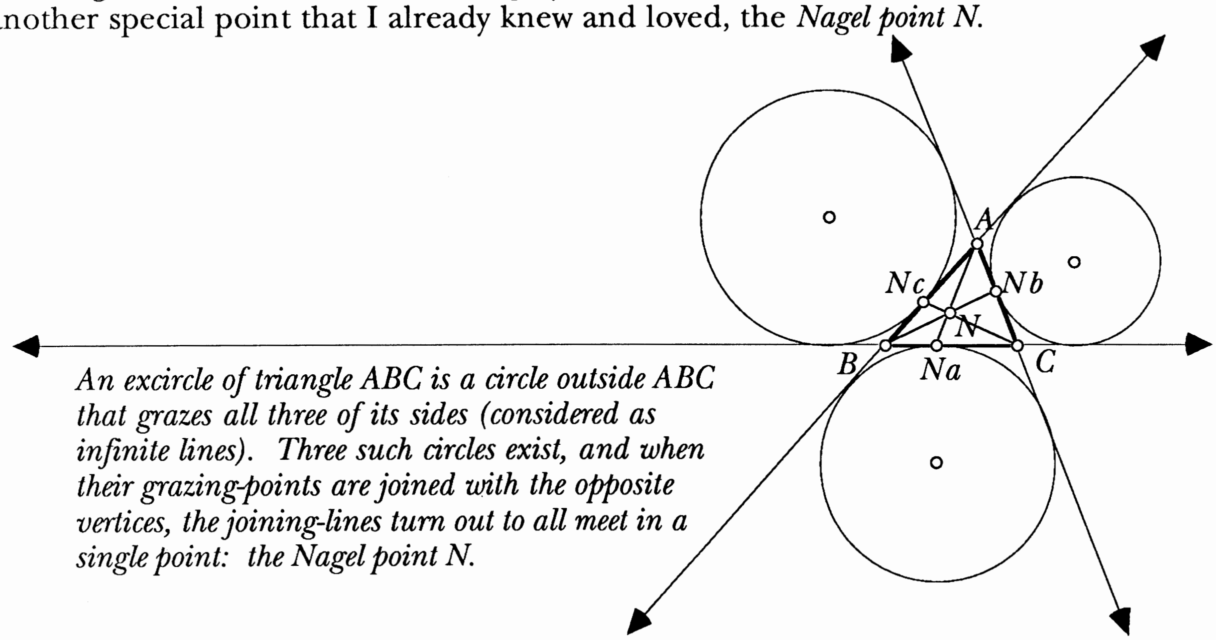
A few of these books referred to a long-out-of-print volume by Julian Lowell Coolidge, called *Treatise on the Circle and the Sphere,* published in 1916. I didn't know whether this book or any of the other old-timers that were occasionally cited would have anything much to say beyond what my modern books collectively had told me, but eventually I decided I had better go check out what my university's math library had on the subject. Soon, I found myself browsing through perhaps the dustiest of all the library's many dusty shelves — those in the old-fashioned-geometry section — and there I came across Coolidge, which I found had lain undisturbed for some 12 years, and for 9 years before that, and then before *that,* yet another 10 years. In other words, it had been checked out only three times since 1960. To my surprise, it was quite a big tome — some 600 pages jam-packed with beautiful diagrams and theorems. A moment's skimming was enough to tell me that this was a treasure trove of geometric gems. Without further ado, I checked it out and joyfully took it home.

Browsing through Coolidge's Chapter One (a small book in itself, rich enough to make me feel very humble), I came across something that almost took my breath away. There was apparently a second segment that not only was *reminiscent* of the Euler segment, but in fact was deeply *analogous* to it.



The anonymous (but herein-dubbed "Nagel") segment, running from the incenter I to the Nagel point N. One-third of the way along, it passes through the centroid G, and halfway along, it passes through the Spieker circle's center S.

This segment, which seemed, strangely, to have no name, ran from the incenter I to another special point that I already knew and loved, the *Nagel point N.*

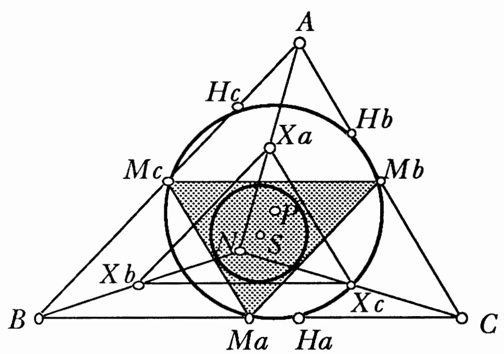


An excircle of triangle ABC is a circle outside ABC that grazes all three of its sides (considered as infinite lines). Three such circles exist, and when their grazing-points are joined with the opposite vertices, the joining-lines turn out to all meet in a single point: the Nagel point N.

Like the Euler segment, the anonymous segment passed through the centroid G as well as through a point S that was the center of another very interesting circle, the *Spieker circle.* Moreover, the relative positions of these points along the segment were the same as in the Euler line: G was one-third of the way from I to N, and S fell exactly halfway between I and N. To cap it all off, Coolidge listed a long and systematic set of parallels between the Spieker circle and the nine-point circle, some of the high points of which are given below (translated into crisper modern terms from Coolidge's slightly verbose "turn-of-the-centurese"):

The nine-point circle...	*The Spieker circle...*
is the circumcircle of ABC's median triangle;	*is the incircle of ABC's median triangle;*
has radius one-half that of ABC's circumcircle;	*has radius one-half that of ABC's incircle;*
is the circumcircle of the triangle whose vertices are the midpoints of the segments linking ABC's vertices with its orthocenter;	*is the incircle of the triangle whose vertices are the midpoints of the segments linking ABC's vertices with its Nagel point;*
passes through the points where ABC's sides are cut by the lines linking ABC's vertices with its orthocenter (i.e., the feet of ABC's altitudes).	*is tangent to the sides of ABC's median triangle where that triangle's sides are cut by the lines linking ABC's vertices with its Nagel point.*

In the figure below, the Spieker and nine-point circles are shown together.



The larger circle, centered on P, is ABC's nine-point circle; the smaller circle, centered on S, is ABC's Spieker circle. The shaded triangle is ABC's median triangle, and the other small triangle is the "auxiliary triangle" belonging to ABC's Nagel point N, whose vertices are halfway between N and points A, B, and C. Note that lines NA, NB, and NC cut the Spieker circle where it is tangent to the median triangle.

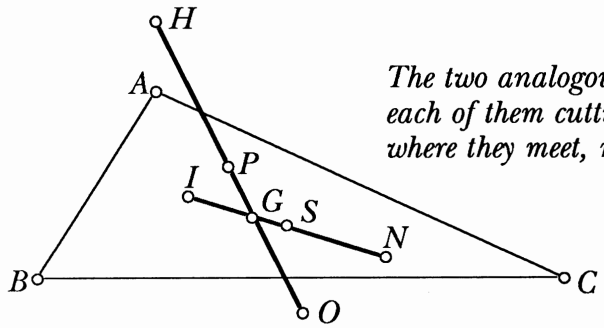
Altogether, this systematic set of correspondences between properties of the Euler and Nagel segments (including the correspondences between properties of the nine-point and Spieker circles) constituted one of the most remarkable and complex mathematical analogies I had ever run across. And to my great pleasure, it restored the honor of the incenter, while also elevating the Nagel point to a level of respect much higher than I had previously accorded it.

I wondered to myself, "Why does this fantastic second segment have no standard name? Why is it not routinely mentioned in the same breath as the Euler segment? Why are the two of them not treated by geometers as precisely equal companions?" Even Coolidge didn't go much beyond the mere act of describing this segment. At the end of Chapter One, he did go so far as to suggest that there probably are other circles analogous to the Spieker circle that remain to be discovered, but that was about it.

Surely, I thought, there is more to it than this. In mathematics, such a striking and intricate analogy can't just happen *by accident*! There's got to be a *reason* for it. But there was no discussion in Coolidge of *why* the parallels were so perfect and so systematic. In the whole library I found only one other book that mentioned this segment, again leaving it nameless and giving less information on it than Coolidge.

I was baffled. Why was this companion segment — which I began calling the

Nagel segment, after the discoverer of its outlier endpoint — so neglected? Was it truly less important than the Euler segment? Or was it just that it had been discovered at a time when people were beginning to lose interest in this kind of geometry? I could not help but mull this over, and the image of these two segments, each one lopsidedly cutting the other into two pieces, reverberated through my head intensely.



The two analogous crisscrossing segments OGPH and IGSN, each of them cutting the other one in the same lopsided way where they meet, namely at ABC's centroid G.

In order to gain a deeper intuitive feel for these things, I ambled into my study, plunked myself down in front of my trusty Macintosh, fired it up, and double-clicked on the icon labeled "Geometer's Sketchpad". Up came a mostly empty screen (representing a blank sheet of paper) with a bar on the left side containing a few icons for tools with which to draw lines, circles, points, and so on. By selecting certain of these icons and then clicking on various spots of the "paper", I had, within a couple of minutes at most, constructed a picture, such as shown above, of a triangle *ABC* with its two associated segments. It looked excellent, but this picture was not the destination — it was just the starting-point.

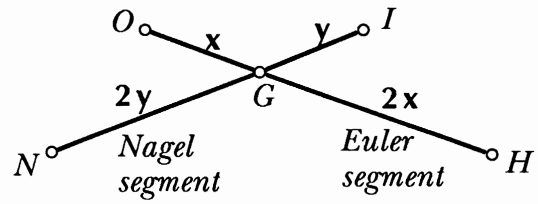
I clicked on point *C* and, with the mouse in my right hand, started "dragging" it around the screen. As I did so, everything else that depended on point *C*, whether point or line, started moving in synchrony, *perfectly maintaining the geometric relationships established by my construction.* In other words, I could now watch the dynamic way in which the Euler and Nagel batons swiveled around simultaneously as the triangle defining them changed. This "dynamogram" was revelatory in a way that no static image could possibly be.

I certainly feel fortunate and grateful to have this wonderful program at my fingertips, and I truly wish some of the old-time geometers, such as Euclid, Apollonius, Pappus, Menelaus, Desargues, Euler, Ceva, Poncelet, Steiner, von Staudt, Brocard, Nagel, Gergonne, Spieker — and let's not forget Coolidge, of course! — could have seen it. I can't help but believe that they all would have flipped out, so to speak — unless, that is, they were as geometrically gifted as James Gleick's recent book *Genius* would have you believe the late great theoretical physicist Richard Feynman was. Here is what Gleick writes:

> In high school he had not solved Euclidean geometry problems by tracking proofs through a logical sequence, step by step. He had manipulated the diagrams in his mind: he anchored some points and let others float, imagined some lines as stiff rods and others as stretchable bands, and let the shapes slide until he could see what the result must be. These mental constructs flowed more freely than any real apparatus could.

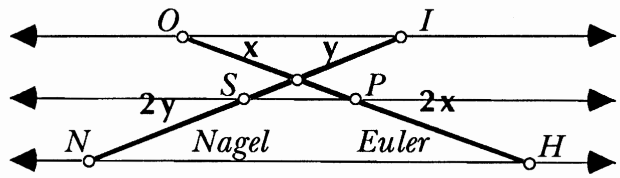
I suppose Feynman could do this to some extent — but then, so can I. For that matter, so can anybody who loves geometry. However, I must say, I am extremely skeptical that Feynman or *anyone* could do anything like what Geometer's Sketchpad allows me to do. The reason is, GS does its arithmetic with something like 19 decimal digits of accuracy, and it can simultaneously move around dozens or even hundreds of complexly interdependent points, lines, and circles (and typical constructions often involve this much stuff). If Feynman could do that, he deserves considerably more than the epithet "genius" — perhaps "paranormal Martian freak" would be more like it. Still, it's a nice story, and Gleick's suggestion of what Feynman could see in his mind does get across a feeling for the immense power one taps into with this program.

So I looked for patterns, questing after comprehensibility. But although my dynamogram was pretty and fascinating, my eyes didn't pick much up at first. Here is a picture of the two crisscrossing segments by themselves, without triangle *ABC*.



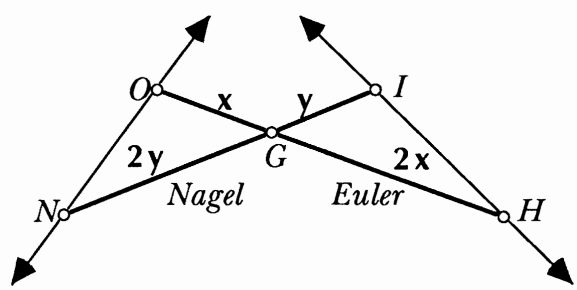
Recall that in the tight analogy between the segments, *O* maps onto *I* and *N* maps onto *H*. It seemed therefore very natural to construct the lines *OI* and *NH*, each of which links counterpart points together. After all, these lines, if built, would constitute a *concrete physical realization* of the abstract analogy — a lovely idea, irresistible to me.

It took but a moment to construct them, and the instant they flicked onto my screen, I saw something most promising — they appeared to be parallel! To test this hopeful hypothesis, all I needed to do was fuss around with the triangle *ABC*, which I immediately did, and I found that no matter how I tweaked it, the *OI–NH* parallelism stayed true. Moreover, I found that the two midpoints, *P* and *S*, when joined, added a *third* parallel line to the first two. I felt as if I had stumbled on a stupendous connection between the segments!



A couple of minutes' thought deflated me rather devastatingly, however. I soon realized that this supposedly "profound insight" was in fact a triviality: *any* two line segments that cut each other in identical length-ratios will always have parallel lines defined by their tips. My parallelism had nothing whatsoever to do with the fact that I was dealing with two Deeply Significant Segments — it was just a simple consequence of similar triangles. I felt ashamed of myself for my premature elation.

The next thing I did was to make the only *other* two lines that remained, *HI* and *NO*. But unlike *OI* and *NH*, these two lines didn't have any strong *reason* pushing for building them.

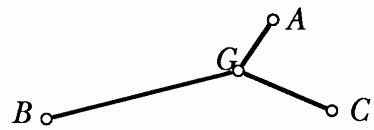


Disappointingly, they turned out to be neither parallel nor perpendicular, and after watching them for a little while, I concluded they were of no interest. But where to look, then, for insight into what lay behind this tantalizing analogy? I was at a loss.

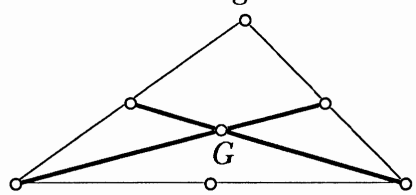
Chapter I
Bedazzled...
by a Sparkling Crystal

Disappointed at having made essentially no progress on the mystery of the interrelationship of the two analogous segments, I gradually started letting the matter slide into "background mode". And so a few days later, I found myself idly thinking about a completely different and rather simple geometric question:

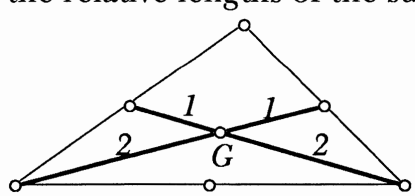
> *Suppose somebody gives me a point G and asks for the general recipe for making triangles having G as centroid. Can I simply "splay out" from G three segments of arbitrary lengths in arbitrary directions, letting their tips define the vertices?*



That, of course, I knew instantly was nonsense. There had to be some *constraints* on the segments that defined the three vertices — after all, there is a *relationship* between a triangle's centroid and its vertices! But what were these constraints? I remembered that the segment from the centroid to any vertex forms a part of one of the triangle's three *medians* (a segment that links a vertex with the opposite side's midpoint). So this told me the constraint on my "centrifugal" segments had to be essentially the same as the relationship that holds among the medians of a given triangle. And how are medians related? I sketched this picture on paper, showing just two medians (it's easier to think about two things than three):

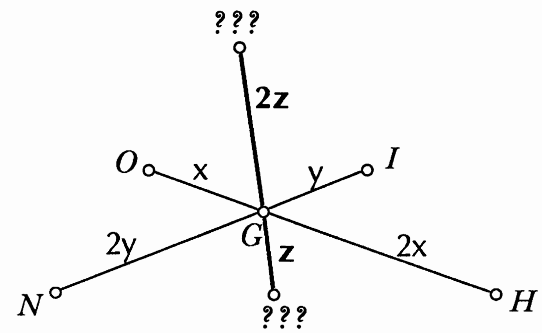


I wanted to see how they constrain each other. Since they, rather than the sides of the triangle, were the focus of my thinking, I went back over the medians a few times in red ink to make them stand out. And then I remembered that medians always cut each other in a characteristic way, namely into subsegments whose length-ratio is 1:2. So I wrote numbers indicating the relative lengths of the subsegments, as follows:

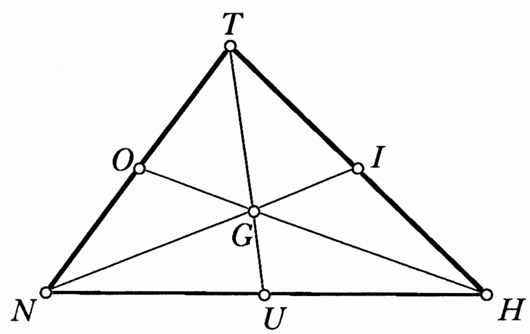


The instant I put those numbers down on the paper, something clicked in my mind. This was a turning point in the whole process, for in the crisscrossing heavy lines of this picture I suddenly recognized something familiar — my two fundamental segments crisscrossing each other, slicing each other up in that lopsided 1:2 way. Unexpectedly, the little puzzle had brought me back to my earlier quest via the back door! *Looking at one picture and seeing another* — that was what everything had hinged on.

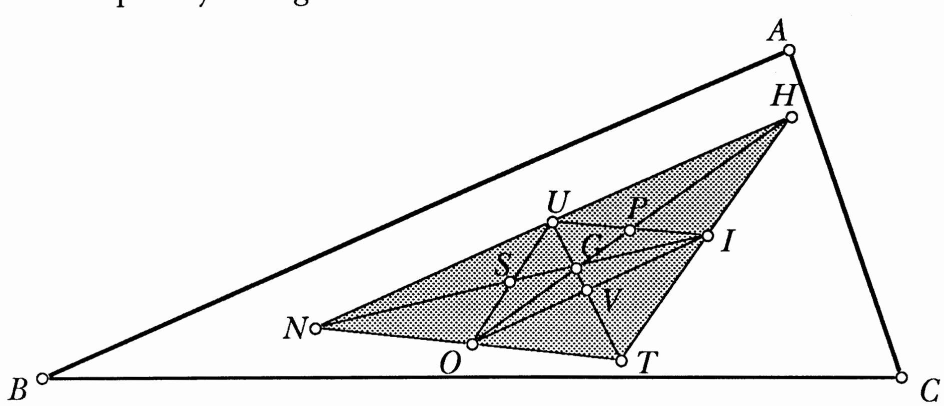
This new vision meant that I could interpret my two segments as *medians of a hidden triangle.* And of course, since any triangle has *three* medians, this meant there was *one more segment,* which would complete the trio of which the Euler and Nagel segments were now seen to be simply the first two. Excitedly, I penned it in:

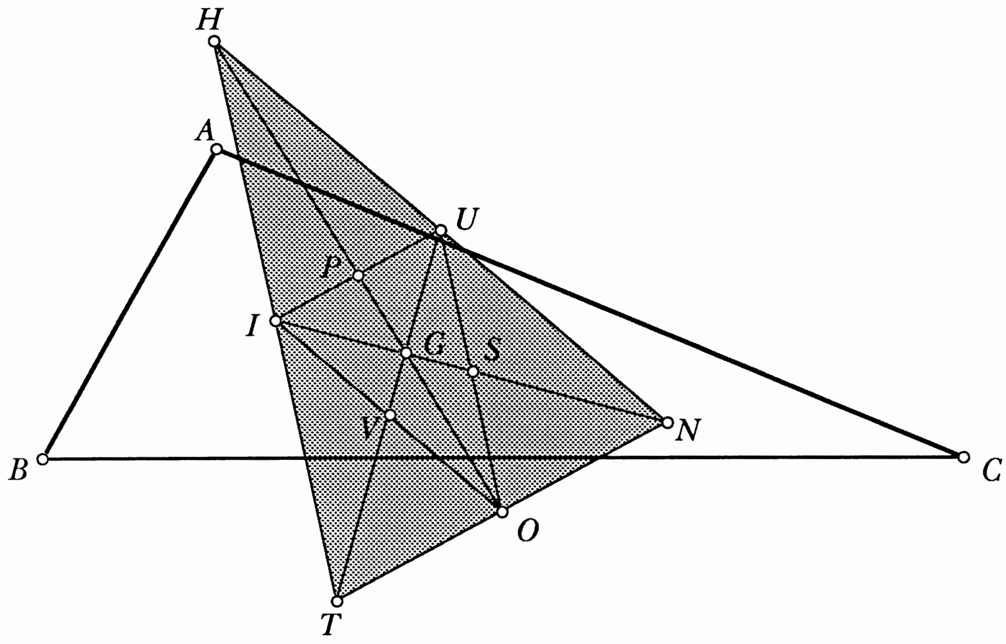


Here we see the third median, also of course divided into subsegments having a 1:2 length-ratio. I suspected that its endpoints were novel points, but then again they might be well-known, so I simply labeled them with question marks. The next step was to draw the full triangle implicitly defined by these three medians. I was starting to tingle now, because it seemed to me that since *two* of the medians were known segments having very important endpoints and midpoints (not to mention sublime properties), it almost *had* to be the case that this third median's endpoints and midpoint would also have absolutely fundamental properties.



Was I on the verge of a significant discovery? It certainly felt that way. My new triangle gave me a strong sense of symmetry and closure — the sense of bringing something beautiful but unfinished to its inevitable completion. Since two of its sides' midpoints were, by chance, named by vowels, I called the new midpoint "*U*", and the new vertex became "*T*". I also gave the name "*V*" to the midpoint of the new median. This point was the counterpart to the nine-point center *P* and the Spieker center *S*. So my new segment — the third median and the analogue of segments *IGSN* and *OGPH* — was *UGVT*. Betraying my rather juvenile excitement, I gave the triangle as a whole the corny name "Magic Triangle". Shown below are the Magic Triangles for two different triangles *ABC*. Of course, these static diagrams are no substitute for a Geometer's Sketchpad dynamogram. Too bad!

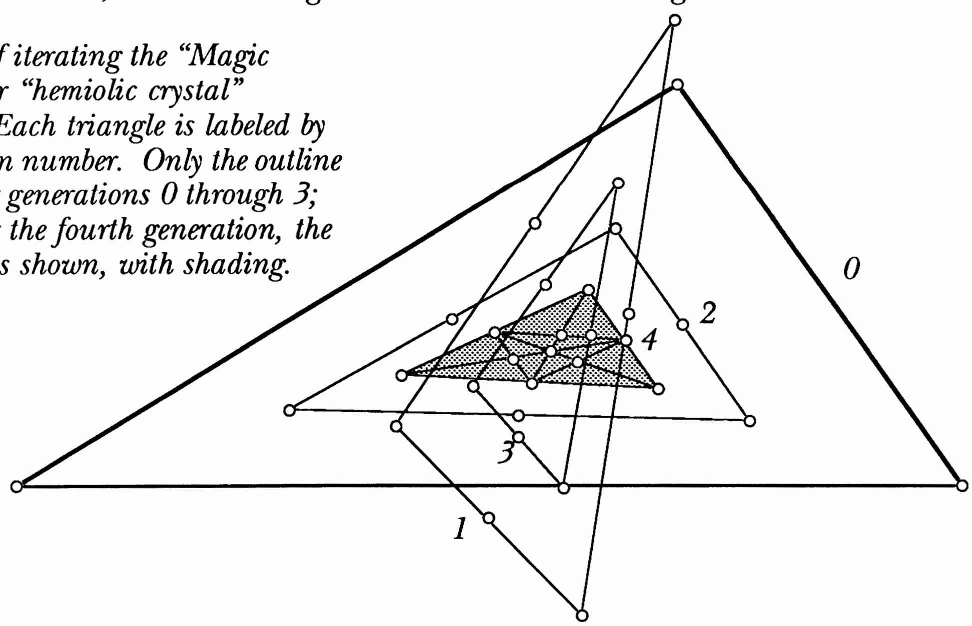




The irony did not escape me that two of my Magic Triangle's sides (*HI* and *NO*) were lines that I had earlier rejected as meaningless. At the earlier time, it had not occurred to me that the lines' *intersection-point* might be of interest. In that context, I simply wouldn't have had any grounds for suspecting that that point might mark the tip of a new segment analogous to the first two. But now, I saw things I never would have seen because I now perceived the two segments *as medians.* I will come back to the cruciality of this idea of "seeing X *as* Y" toward the end of this essay.

An idea that immediately aroused my curiosity was: What happens if you take the Magic Triangle of the Magic Triangle of *ABC*? What happens if you iterate the operation *n* times? I explored this and soon learned that in general there are no simple patterns. However, there are some exceptional pretty cases, such as the one shown below, where there is a kind of zooming-in effect, with the even generations all resembling each other, and the odd generations all resembling each other.

The result of iterating the "Magic Triangle" or "hemiolic crystal" operation. Each triangle is labeled by its generation number. Only the outline is shown for generations 0 through 3; however, for the fourth generation, the full crystal is shown, with shading.



At about this point, I started feeling a little self-conscious about the hokey name "Magic Triangle", and I found a more poetic name that seemed to say a little more and to fit very nicely: ***hemiolic crystal.*** With its crisscrossing medians, its six external points, and its four internal points, it looked at least a *little* bit like a crystal. The term "hemiolic" came from the musical notion of *hemiola,* which refers to the ambiguity inherent in a six-beat measure: should it be heard as *three* groups of *two* notes each, or as *two* groups of *three* notes each? On the one hand, this is nothing but a fancy name for the hardly-earth-shattering fact that multiplication of two particular small integers is commutative; on the other hand, when one hears a given melody grouped in these two opposing manners, it is a striking perceptual shift. Some beautiful musical effects are based on hemiola, and I felt it was a very appealing concept to bring in. For example, should we think of the six points forming the exterior of this crystal as *two sets of three points* (*HNT* and *OIU*)? Or should we rather think of them as *three sets of two points* (*OH, IN,* and *UT*)? The possibilities implied by these two complementary views seemed pregnant with meaning, and so "hemiolic crystal" it was.

But needless to say, all this excitement was predicated on something entirely unsure: that my new third segment was in some sense *meaningful.* I was convinced it would have to be, but I still didn't know the first thing about its new points *T, U,* and *V*! Nonetheless, I had inner confidence, perhaps based, unconsciously or semiconsciously, on considerations of the following sort. I could imagine three possibilities regarding *T, U,* and *V*:

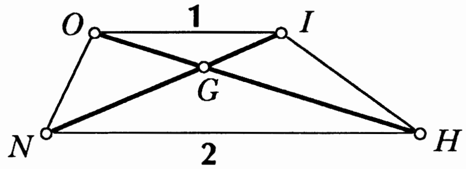
(1) *All three* of my supposedly new points *T, U,* and *V* turn out to be known entities. If this is the case, fine — because I would have revealed an unsuspected unity among them (the fact that they belong to a third segment much like the Euler and Nagel segments). That would be a lovely new discovery — and of course the harmony among the *trio* of segments constitutes a yet higher-level unity, a fact that, it seemed to me, would constitute a quite fundamental and respectable discovery.

(2) *At least one* of my new points *T, U,* and *V* is a new discovery and turns out to have interesting and novel properties. This is even better than (1), because then not only have I discovered a sublime high-level trio and a great new segment, but also one or more nifty new special points!

(3) *None* of my new points *T, U,* and *V* has any interesting property at all. On one level, this would seem pretty dismal, but just think — on another level, the absence of meaning would be so strange that it would in itself be fascinating. How amazing that two fundamental segments define a third segment in an utterly natural and closure-creating manner, and yet the third segment turns out to be meaningless! This would be a sort of paradoxical twist or surreal joke, a bit like the old math-joke that "proves" the nonexistence of "uninteresting numbers". The proof goes like this: if there were uninteresting numbers, the smallest of them would be highly *interesting,* for that very reason. Hence there can be no *smallest* uninteresting number, ergo no uninteresting number at all — QED. However, it seemed to me that the paradox of "uninterestingness as a source of interest" would apply even more to the segment than to the numbers. On the other hand, this whole possibility seemed so unlikely as to hardly merit consideration.

All in all, then, I felt secure no matter how things might turn out — provided, that is, that the whole idea hadn't already been discovered, in which case I would just be a jackie-come-lately. Since this kind of letdown had already happened a couple of times during the previous months of my geometry binge, I was fairly used to the feeling of disappointment. But somehow this case felt quite different. This time, I thought, I have really found something of my own!

Certainly Coolidge hadn't known about the new segment or the crystal in 1916, for his book doesn't mention either one at all. In fact, though, he came pretty close to finding them — he explicitly talks about the "trapezoid" defined by the points *OIHN*, but he doesn't see that it naturally beckons one to *complete* it by putting a little triangle on its short top-side.



And Roger Johnson, a disciple of Coolidge's who came out with his own scholarly treatise on circles and triangles in 1929, briefly describes the unnamed second segment, but doesn't mention any connections between it and the Euler segment. So at least up till 1929, my crystal was certainly completely unsuspected. "And," I thought to myself, "how much deep exploring of triangles has there been since 1929?" Not too long thereafter, people began turning away from such "simple-minded" and concrete matters, and soon most mathematicians were swept away by the tides of Bourbaki-ism and other such abstraction-favoring schools of thought.

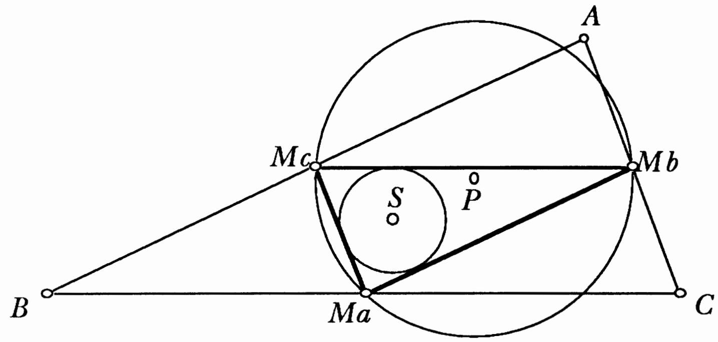
Later, I found a more recent book by Nathan Altshiller-Court — *College Geometry: An Introduction to the Modern Geometry of the Triangle and the Circle,* published in 1952. This extremely comprehensive treatise mentioned the Nagel segment (again giving it no name and according it much less honor and space than its "identical twin", the Euler segment) but didn't connect it with the Euler segment, much less hint at the existence of a third, related segment.

All this gave me quite a bit of hope that I was the first to see this gem. Was I truly the first person in decades to find a major new property of the triangle? In an exuberant mood that evening, I described my breakthrough to my wife, and she fantasized that now, some 25 years after having abandoned mathematics, I might be awarded the Fields Medal! We chuckled over this idea, especially the irony that I had quit math in utter despair over its enormous abstraction. Of course, any talk of a Fields Medal was just a silly dream and we knew it, but it did occur to me to wonder just *why* it was so utterly laughable to think that work in Euclidean geometry, no matter *how* elegant or new or fundamental, might be considered worthy of the Fields Medal — or even worthy of serious attention by mathematicians at all.

But let me not get ahead of my story, for such meta-level musings are the focus of the next chapter. To conclude this chapter, then, I would like to tell about the rest of the discovery: the search for the meaning of *T, U,* and *V.*

The day after I had come up with the notion of the crystal itself, I made an all-out effort to find its meaning — that is, to "decipher" its new points. My first attempt focused on *V,* because it seemed the easiest. My idea was that, just like *P* and *S,* its counterparts on the other two medians, *V* ought to be the center of some spectacular circle that managed to jump through several hoops at once. So on the screen I drew a circle of variable size centered on *V* and let it shrink and grow while carefully looking for coincidences of any sort — simultaneous tangencies here and there, simultaneous passages through various points, simultaneous whatever! But I saw nothing. It was somewhat shocking, but I took it in stride. After all, it would be nice if *V* had beautiful properties but ones so subtle that nobody before me had ever noticed them. I still had high hopes that *V* would in the end turn out to have lots of meanings.

As a result of this mild setback, I went back to thinking a bit more about *P* and *S,* since it was on them that my analogy, for whatever it was worth, was based. On reflection, it seemed to me that both *P* and *S* really belonged more to *ABC*'s *median triangle* (the triangle formed by the midpoints of *ABC*'s sides) than to *ABC* itself, because they are the centers of that triangle's circumcircle and incircle, respectively. Semi-symbolically stated, P is the O of the median triangle, and S is the I of the median triangle.



MaMbMc is ABC's median triangle. The circumcircle of this little triangle is the nine-point circle of ABC itself, and similarly the incircle of this little triangle is the Spieker circle of ABC itself. Therefore, P is the O, and S the I, of the median triangle.

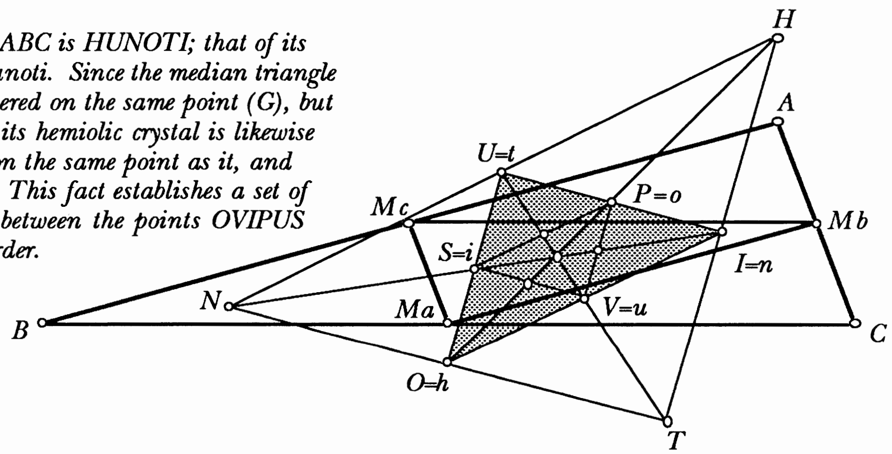
When I thought about this in connection with the medians of my crystal, the analogy seemed clearly to be telling me this: "*V* is the *U* of the median triangle." But what did this mean? It seemed to me to suggest that the meaning of *V* would be elucidated only after I elucidated the meaning of *U*, and so this pushed me to switch the focus of my quest to the two new points directly *on* the hemiolic crystal (*T* and *U*), rather than the single new point *inside* it.

I made a small and rather simple discovery at this point, which was that by drawing the segments connecting *O*, *I*, and *U* inside the hemiolic crystal of *ABC*, I was thereby constructing the hemiolic crystal of *ABC*'s median triangle. In rather opaque language, I had found the following result:

> ***OVIPUS–hunoti Theorem.*** The hemiolic crystal of *ABC*'s median triangle is the median triangle of *ABC*'s hemiolic crystal.

It sounds kind of grandiose, but really it is quite trivial. Pictorially, it looks like this:

The hemiolic crystal of triangle ABC is HUNOTI; that of its median triangle MaMbMc is hunoti. Since the median triangle is half the size of ABC and centered on the same point (G), but rotated 180° with respect to it, its hemiolic crystal is likewise half the size of HNT, centered on the same point as it, and rotated 180° with respect to it. This fact establishes a set of six one-to-one correspondences between the points OVIPUS and the points hunoti, in that order.



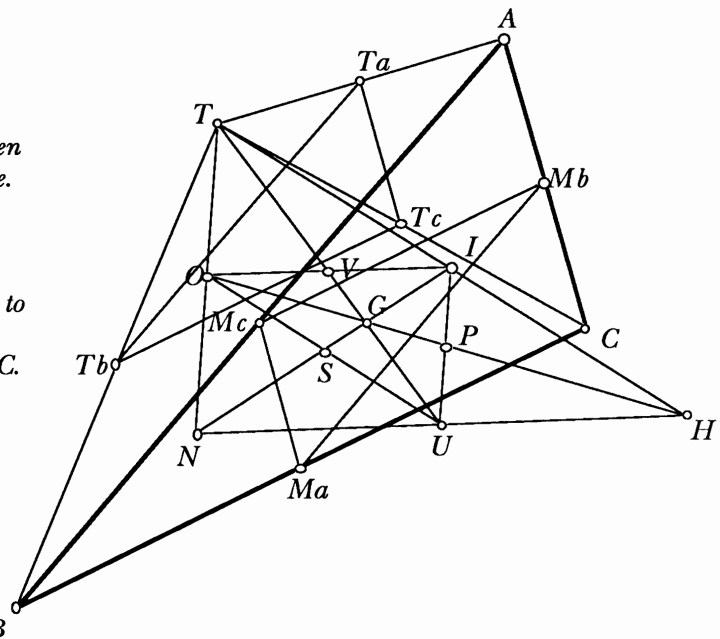
At this point, I went off on a long and completely fruitless wild-goose chase, motivated by a very jumbled-up analogy that I made in a confused moment, involving my new points *T* and *U* and several of the old points of the hemiolic crystal. The details of my confusion of course don't matter, so I won't repeat them here. What *does* matter is that at the end of this wasted time I got extremely discouraged about the probability of my crystal's meaningfulness — so discouraged, in fact, that I was even led to questioning whether I had correctly understood the passage in Coolidge where I had learned about the so-called "Nagel segment". I started wondering if the whole idea of the hemiolic crystal hadn't perhaps been a hallucination caused by a misunderstanding! Maybe the Nagel segment involved some *other* points, after all, or maybe it wasn't really as analogous to the Euler segment as I had at first thought.

Maybe my whole beautiful dream was about to go up in smoke.

So I went back to Coolidge and rechecked the passage in which he describes the anonymous Euler-like segment. To my relief, I found I had gotten it exactly right. I hadn't misunderstood anything. But my sense of self-confidence had been quite shaken by my stupid false analogy, and I felt I needed some outside guidance. Since I was already looking at the crucial section of Coolidge, I let it serve that role. Very carefully, I hunted through it for any possible hints of things to look for in old or new dynamograms. It was then that I first noticed the rather important role played by two "auxiliary triangles" — one involving *ABC*'s orthocenter *H*, the other involving *ABC*'s Nagel point *N*. Each auxiliary triangle was constructed in the same way. You took a point *X* (*H* in one case and *N* in the other), connected it in turn with *A*, *B*, and *C*, and then bisected those segments. The three midpoints defined *X*'s auxiliary triangle.

Coolidge was whispering a secret, but to my ears his analogical message was loud and clear: *Construct the auxiliary triangle belonging to T!* Clearly, it was time to wake up Geometer's Sketchpad again. I first constructed a triangle *ABC* and its hemiolic crystal *HUNOTI*, then the auxiliary triangle belonging to *T*. Of course, the intricate interdependencies of all these points and lines were perfectly maintained no matter how I distorted the original triangle *ABC*, so I knew I had front-row seats to whatever interesting phenomena there might be — if there were any to be found, that is!

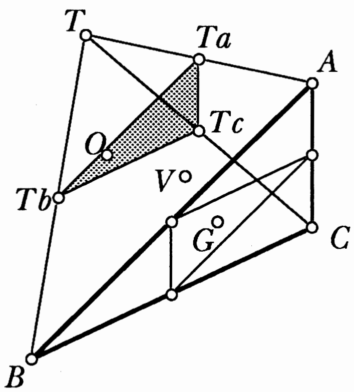
A very cluttered picture. Triangle ABC is given at the outset. MaMbMc is its median triangle. HNT are the vertices of ABC's hemiolic crystal, with OIU being the midpoints of its sides and PSVG its four interior points. TaTbTc is the "auxiliary triangle" belonging to point T — that is, the triangle whose vertices lie halfway between T and the vertices of ABC. It is hard to notice any relationships in this complex an image, even if it is a dynamic image on one's screen.



In its original form, this picture was very cluttered. I couldn't make head or tail of what was going on, there were so many extraneous points and lines around. The fact that it was dynamic didn't help at all. It seemed I would simply have to do some judicious pruning — namely, I would have to "hide" a bunch of points and lines. This, incidentally, is one of the many attractive design-features of Geometer's Sketchpad. To make almost any dynamogram of interest requires a good number of construction lines, and they often clutter up the picture enormously. No problem: any object can be made invisible by "hiding" it. Its invisibility will not interfere at all with the interrelationships it enjoys with other objects in the diagram. Once things are hidden, as you move any base point around, its dependent points and lines will swing around each other in subtle trajectories for seemingly magical reasons! Hiding objects is a little bit like a stage trick, in which invisible props allow magical things to happen — for example, hiding the wires on which Mary Martin is suspended, making it seem as though Peter Pan is flying unsupported across the stage.

So, I started stripping away line after line, point after point, in an attempt to boil the picture down to its essence, fearing all the while that I might well be throwing the baby out with the bathwater. Still, I knew that I would never be able to *find* the baby if I didn't do this, so I pared the image down, very gradually, winnowing out a point here, a line there, until it started seeming that I had reached roughly the right level of simplicity. I also highlighted *T*'s auxiliary triangle by shading it, since that was supposed to be the real focus of my perception.

The previous diagram after drastic simplification, and slight alteration of the triangle ABC. Most of the points and lines of ABC's hemiolic crystal have been made invisible. Only T, O, V, and G are left. T's auxiliary triangle has been shaded for emphasis. All of a sudden, there is a strong suggestion of three-dimensionality.



At this stage, something leapt out at my eye — namely, the undeniable *three-dimensionality* of the picture. *T* and its auxiliary triangle seemed to form a tetrahedron with *ABC* as its base. "Aha — this could be the key to it all!", I thought. But then, once again, that familiar feeling of foolishness washed over me, as it hit me that this would hold no matter what point's auxiliary triangle I made. This property had nothing to do with the fact that I was dealing with a special point *T*. Once again, I had experienced a "false epiphany", showing what a naïve geometer I really was.

I felt quite humiliated, but even so, I wasn't daunted. I just kept on looking for something interesting. And as I randomly dragged a vertex of *ABC* around, a small coincidence caught my eye — the fact that point *O* kept staying inside the shaded region. It seemed inconsequential, but I wondered, "Is this really *always* so? Why would that be the case?" So I kept on watching it as *ABC* changed shape, and indeed it never left the shaded region. I noticed that whenever *TaTbTc* got long and pointy, *O* seemed to cling very close to one of the longer sides, and to slide down toward the pointy end. This rang a bell. I *recognized* this trait — it's a characteristic of the Nagel point of a triangle! (How did I know this? Easy — I had spent some time watching Nagel points sliding around inside their home triangles, during an earlier stage of my geometry binge.) At last, I was picking up a meaningful message: *O* seemed to be acting like the Nagel point of *T*'s auxiliary triangle.

Of course, this needed some confirmation. To make sure that my intuition was right, I quickly added some new construction lines to my dynamogram, which defined the *actual* Nagel point of *T*'s auxiliary triangle. This new point, *N'*, landed smack on top of *O*, and as I moved things around, their co-incidence never changed at all. I was in like Flynn!

For a moment, I was a bit worried that this, too, might prove to be yet another false epiphany — another piece of geometrical trivia — but as I considered it carefully, I came to the firm conclusion that this was a meaningful and unexpected finding. So now I knew at least *something* definite and new about *T*:

Triangle ABC's circumcenter O is the Nagel point of the auxiliary triangle of T.

It wasn't exactly the most perspicuous and exciting property of all time, but it was at least *something*, and I felt that I now held the key to the unraveling of the properties of *T*, and maybe also of *U*.

Perhaps most importantly, I felt that I now had genuine confirmation that the new points of the hemiolic crystal were *not meaningless.* My initial intuitive sense that in making my new third segment, I was "onto something" had been upheld, and my fears that I was a dumbbell who had simply misread Coolidge vanished into thin air. This was enough for one day, and I went to bed with a feeling of great satisfaction.

The next morning, however, I felt distinctly uneasy with the current stage of my discovery. For a segment that was supposed to be on a par in importance with the Euler segment, my new segment was surely not *acting* very important. The only piece of information I had about it was a rather obscure fact about one of its endpoints. Surely, there must be more to the new points than just that one teeny theorem!

By the way, note that I just referred to my screen-based observation as a "fact" and a "theorem". Now any red-blooded mathematician would start screaming bloody murder at me for referring to a "fact" or "theorem" that I had not *proved.* But that is not my attitude at all, and never has been. To me, this result was so clearly true that I didn't have the slightest doubt about it. I didn't *need* a proof. If this sounds arrogant, let me explain. The beauty of Geometer's Sketchpad is that it allows you to instantly discover if a conjecture is right or wrong — if it's wrong, it will be immediately obvious when you play around with a construction dynamically on the screen. If it's right, things will "stay in synch" right on the button no matter how you play with the figure. The degree of certainty and confidence that this gives is downright amazing. It's not a proof, of course, but in some sense, I would argue, this kind of direct contact with the phenomenon is *even more convincing than a proof,* because you really see it all happening right there before your eyes. Seeing is believing, as they say, and for me there is no clearer illustration of this homily than the experience of playing with Geometer's Sketchpad.

None of this means that I did not *want* a proof. In the end, proofs are critical ingredients of mathematical knowledge, and I like them as much as anyone else does. I just am not one who believes that certainty can come *only* from proofs. When in my distant math-major past, I made various mathematical discoveries, I was almost always completely certain of their truth long before finding any proof. In this particular case, even if my little result wasn't yet deserving of the title "theorem", it was, as far as I was concerned, a *fact.*

In any case, as I lay in bed, musing over my partial success, I thought that my isolated little result was rather pathetic, and that surely it had to be accompanied by more. My unproven "theorem" could be tersely phrased as follows, using an obvious notation to symbolize the notion of "auxiliary triangle":

O is the N-point of the T-Δ.

Sometimes I even wrote it in this highly compressed manner:

O is the N of T.

This most compact way of writing it highlighted the letters "O", "N", and "T", and that in turn made me think about where those three points fit into the crystal *HUNOTI.* I noticed that they all lie on *NOT,* one of its three sides. But since my feeling about the crystal was that it is a highly symmetric structure, it seemed natural to wonder whether *analogous* statements might not hold for the other two sides — *HUN* and *TIH.* Thus I was led, purely by a sense of elegance, analogy, and symmetry, to make the following two rather bold speculations:

U is the H-point of the N-Δ.

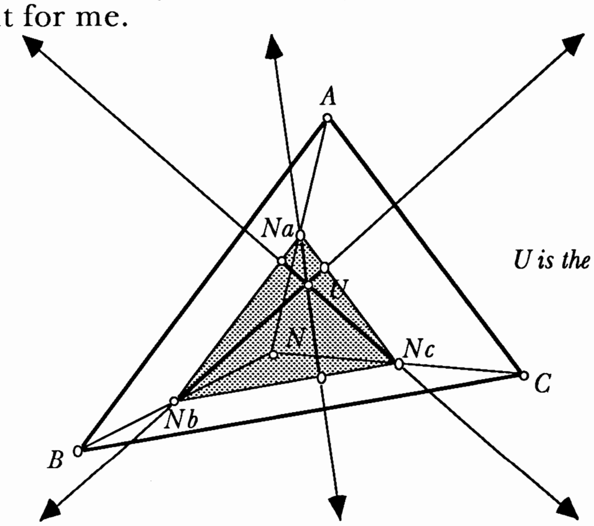
I is the T-point of the H-Δ.

I didn't have a great deal of confidence in either of them, because although *formally* they made a symmetric trio with the first one, when you looked at their *meanings,* they said amazingly different things. Here are all three, spelled out more completely:

(1) O, the circumcenter of ABC, is the Nagel point of T's auxiliary triangle.
(2) U is the orthocenter of the auxiliary triangle belonging to ABC's Nagel point.
(3) I, ABC's incenter, plays the T role for the auxiliary triangle of ABC's orthocenter H.

These statements seem to have nothing to do with one another! They involve completely different concepts — it's simply that those concepts have been cyclically permuted from one line to the next. The only reason to call these statements "analogous" is because they all have the common form "*X* is the *Y* of *Z*'s auxiliary triangle", and because each one is formally associated with one of the three sides of the hemiolic crystal — a pretty dubious basis on which to bank. And I certainly wasn't holding my breath. Still, I felt there was at least a sporting chance that this guess might pan out, so I pitter-patted down the hall in my bedroom slippers, turned on the old Mac, and clicked on my faithful "verification engine", Geometer's Sketchpad.

In the twinkling of an eye, I had made a new dynamogram showing *ABC,* its Nagel point *N,* and its *U* point. I then constructed *N*'s auxiliary triangle. The question was, did *U* look like that triangle's orthocenter? It looked at least *plausible* on the screen, but I needed *proof* — "eyeball proof", that is. And so I constructed the three altitudes of *N*'s auxiliary triangle. The first one ran straight through *U,* skewering it perfectly. So did the second, and so did the third. Bingo! Moreover, when I moved the vertices of *ABC,* everything stayed completely right on target. There was no doubt that the first of my two analogy-based speculations was true! It was quite a stunning moment for me.

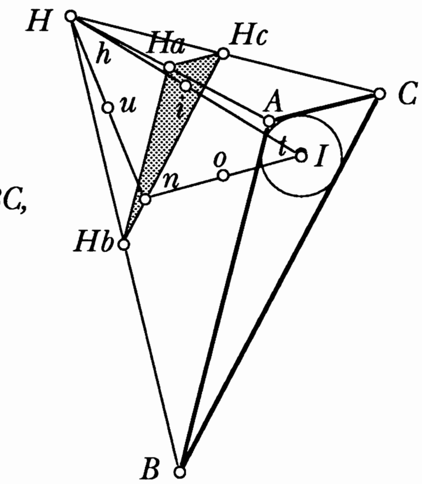


Eureka!
U is the H of N's auxiliary triangle!

The question remained, what about the *other* speculation? Now that *one* of them had been verified and was known to be true (*pace* all red-blooded mathematicians!), I was willing to bet high stakes on the other one. In fact, my mind would have been totally boggled if the last one hadn't been true as well. After all, it closed the *HUNOTI* circle, so to speak.

However, I wasn't going to deny myself the pleasure of actually *seeing* its truth on the screen. Nor did I have so much chutzpah that I could simply nonchalantly skip the act of perceptual verification, relying instead purely on *intellectual* knowledge (or faith, if you insist) that it must be true. I have far too concrete a mind to do that. So I made the dynamogram, and played around with it. To my astonishment, the two points that were supposed to be coincident with each other were moving around completely independently of each other. No relationship at all! Could it be that my third and final statement was *wrong* while the other two were *right*?

I had a moment or two of self-doubt, but then my ever-strong belief in beauty and symmetry — a kind of "inner compass" — resurged and started to gain the upper hand. I thought to myself, "Don't be silly. The theorem is *of course* true. It follows, therefore, that I've simply *made a mistake* somewhere in this construction. Let me try again." And in trying it again, I discovered the slight error I'd committed. I was right that I'd been wrong. And indeed, the correctly-built new dynamogram fully verified the third member of the trio, and victory was mine.



Showing that I, the incenter of ABC, plays the T role in the hemiolic crystal hunoti belonging to H's auxiliary triangle HaHbHc.

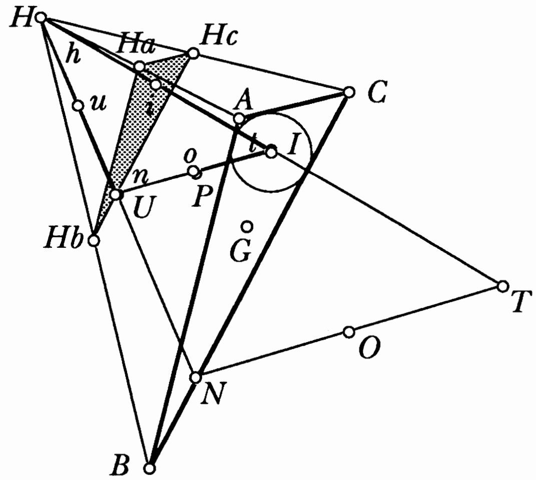
It strikes me that a key ingredient here was the fact that I had enough self-confidence to trust my "inner compass" more than what I saw before me on the screen. If I hadn't, I might well have been stopped in my tracks, and never made the third discovery that closed the circle — or the triangle, to be more accurate.

Given the way the statements associated with the three sides of the crystal all "chain" or "dovetail" together, I decided to call my cyclic trio of results the "Garland Theorem", as it represents a "garland" of linked results. I was now extremely satisfied: I had found enough to convince me that the hemiolic crystal was a nontrivial new idea with at least a few elegant and nonintuitive (or non-obvious) properties. So, at the conclusion of my second day, this was how things stood.

The next morning, I felt far less satisfied. Just as on the previous morning, I felt that what I had found so far was really *not* very impressive, if I was indeed dealing with something whose importance was on the level of the Euler segment. What I had found was just a set of three little curiosities. Big deal! I needed something much more impressive. I also felt I needed proofs. Dynamic visual verifications were fine for just me, but if I wanted to show this stuff to other people, especially professional mathematicians, I would be laughed off the stage if I had no proofs!

Never having been the completely analytic do-it-in-your-head type, I went back to Geometer's Sketchpad, to be in contact with the phenomena themselves in a very concrete way. I had a vague hunch, from things I had seen on the screen the day before, that there were *more* results to be found along the same lines as the three cyclic components of the Garland Theorem. The obvious thing to do, it seemed, was to take an auxiliary triangle such as *H*'s, and to take *ABC*, and to exhibit their *full* hemiolic crystals together on the screen. To be sure, this might make a hugely messy screenful, but then again, it might not. No harm trying. And here's what I found:

ABC's hemiolic crystal HUNOTI was constructed first. Then H's auxiliary triangle HaHbHc was constructed (shaded), and finally its hemiolic crystal hunoti. The two crystals share one vertex and nest in a neat manner, setting up a family of equivalences: H is the H of H; U is the N of H; I is the T of H; and P is the O of H.



The diagram was surprisingly unmessy — in fact, as simple as it could possibly have been. First of all, *ABC*'s orthocenter *H* coincides with the orthocenter *h* of *H*'s auxiliary triangle (this is provable in a snap). This, together with the obvious fact that the auxiliary triangle is half the size of *ABC* and oriented parallel to it, implied that the two hemiolic crystals, one big and one small, must fit snugly together, their sides perfectly superimposing. Thus two of the midpoints of the big one had to coincide with two vertices of the small one, and this implied three further point coincidences. Rather than being baffling, it all made sense in a very straightforward manner. So not only had I now discovered some new facts, I also had a clear understanding of *why* these facts were true. In other words, as new results started pouring in, so did proofs of them! I couldn't have asked for more.

Two more diagrams — one involving the auxiliary triangle of *N* and one involving the auxiliary triangle of *H* — gave me precisely analogous results, and for precisely analogous reasons: the snug nesting of a half-size triangle inside the full-size one. Since each diagram gave me four results where the day before I had had just one, I had almost effortlessly multiplied the content of the Garland Theorem by a factor of four, so that it now ran this way:

The "Garland" Theorem. For any point *X*, define *X*'s *auxiliary triangle,* denoted "*X*-Δ", by joining *X* with each of *A, B,* and *C,* and connecting the midpoints of those segments. The following relations then hold among the points of *ABC*'s hemiolic crystal:

(1*h*) *H* is the *H*-point of the *H*-Δ.
(1*n*) *N* is the *N*-point of the *N*-Δ.
(1*t*) *T* is the *T*-point of the *T*-Δ.

(2*h*) *U* is both the *H*-point of the *N*-Δ and the *N*-point of the *H*-Δ.
(2*n*) *O* is both the *N*-point of the *T*-Δ and the *T*-point of the *N*-Δ.
(2*t*) *I* is both the *T*-point of the *H*-Δ and the *H*-point of the *T*-Δ.

(3*h*) *P* is the *O*-point of the *H*-Δ.
(3*n*) *S* is the *I*-point of the *N*-Δ.
(3*t*) *V* is the *U*-point of the *T*-Δ.

And its proof was now clear, to boot. I was getting to the point where it felt like the hemiolic crystal really was a significant new idea in the geometry of the triangle. Even if my discovery turned out not to be new, these two days had been without a doubt the most exhilarating mathematical experience I had had in 30 years. I was truly thrilled — nay, *bedazzled* — by the hemiolic crystal I had unearthed. And I had a hunch that what I had seen so far was still just the tip of the iceberg.

Chapter U
Bewildered... by the Meaning of it all

The next morning, I was certainly on a high. I had found an elegant new concept, a host of intriguing results associated with it, and even their proofs had come along for free. All that remained to find out was: *Was my idea original?* This question nagged at me all day, and eventually I decided I would at least make an attempt to look in the math library for books or journals that might mention results of this kind.

That evening I went over there, and my first stop was back at the geometry area of the stacks. I carefully hunted for books that dealt with plane geometry, and although I found many, they were all of roughly the same vintage: turn-of-the-century, give or take 20 years. Since I knew Roger Johnson's 1929 volume was probably the most authoritative of the bunch, I felt pretty confident that my result was in none of these. The rest of the books were about non-Euclidean geometry, projective geometry, algebraic geometry, differential geometry, and so on. A fair number of them were so abstruse that they did not deign to mention such pedestrian items as triangles. So once again, but for a very different reason, I was "safe".

My next stop was the journals section. I scoured the several hundred titles for ones related to geometry, and the most appropriate one seemed the straightforwardly titled *Journal of Geometry*. However, upon opening it, I found its contents anything but straightforward. About many of the articles, I had to ask myself, "*This* is *geometry*?!" Here are a few titles, just so you can get the idea:

"Metrizations of orthogonality and characterizations of inner-product spaces"
"Strongly distributive multiplicative hyperrings"
"On automorphisms of *n*-dimensional Laguerre space"
"A group-theoretic characterization of finite derivable nets"
"Semifield skeletons of conical flocks"
"Chain geometries over local alternative algebras"
"Quasi-ordered Desarguesian affine spaces"
"Matroidal hypervector spaces"

Another perspective on modern aspects of geometry is afforded by some theorems. Here are a few memorable gems I collected from articles in the *Journal of Geometry*:

"A complete, convex, externally convex metric space in which metric pythagorean orthogonality is homogeneous is a real inner product space."

"Let π be a semifield plane of flock type and odd order q^2. Then the planes of the skeleton are all semifield planes if and only if π is the Knuth semifield plane of flock type."

"Any order compatible place λ of a sub(skew)field of the kernel of an ordered quasifield $(Q, +, \bullet, P)$ extends to an order compatible place ξ of Q with $A_\xi = \mathrm{conv}(A_\lambda)$."

"The elation group Δ of a finite elation Laguerre plane $\mathcal{L}$ is an elementary abelian *p*-group, each derived affine plane is a dual translation plane such that all dual translations are induced by Δ, and the order n of $\mathcal{L}$ is a power of p."

"A desarguesian projective Hjelmslev plane $\mathfrak{H}(\mathfrak{R})$ over a PH-ring $\mathfrak{R}$ is of level n if and only if the Jacobson radical $\mathfrak{J}$ of $\mathfrak{R}$ is nilpotent of degree n."

"The class of all saturated mixed extensions of linear symmetric designs coincides with the set of all finite Möbius planes."

Perhaps even more telling than this set of prickly titles and forbidding theorems is the statistics I collected on a few issues of this leading journal. I was struck by the fact that most articles seemed to have no pictures at all, so I decided to tally up the picture density. I took three successive issues of the *Journal of Geometry* that appeared in 1991 and 1992. Altogether, they contained 52 articles. Of these, only 13 contained any pictures. The page-level statistics are even more revealing. In these issues there were 602 total pages, but only 39 of them had any pictures! In other words, on the average, 75 percent of the articles (39/52) and 93 percent of the pages (563/602) in the *Journal of Geometry* are pictureless. By contrast, Coxeter and Greitzer's book *Geometry Revisited,* which has 153 pages of text, has roughly 160 separate diagrams — an average of over one per page! (To be frank, I must confess that Coolidge is disappointing on this score. His first chapter does rather well, but thereafter pictures are extremely sparse.)

One might think that the absence of pictures is due to the extreme abstraction of the ideas under discussion, and that there simply *are* no appropriate types of figure that can be drawn at all. One reply would be to ask whether, in that case, one is still really dealing with geometry. Perhaps a new name is needed. Another reply is that many academic people — regrettably many — take actual *pleasure* in being formal and opaque. Perhaps they like the fact of joining a tiny elite clique of co-understanders, or perhaps they simply enjoy jargon for its own sake.

It is precisely this kind of thing that made me throw up my hands in Berkeley in 1967 and drop forever out of mathematics. I had just taken a graduate seminar entitled "Number Theory" in which it turned out that the natural numbers 0, 1, 2, 3, and so on (after which number theory is named) were trotted out only on rare occasions as "trivial examples" of the results being discussed. I simply couldn't face the thought of "number theory without numbers" for the rest of my life. And now, looking through the *Journal of Geometry* some 25 years later, I felt as if I were experiencing a replay of that old experience, simply with numbers replaced by geometric objects like triangles and circles. Who cares about geometry without geometric objects? It would be like the "fine romance — with no kisses" in the old Jerome Kern/Dorothy Fields song — and as a Gershwin song from the same epoch says, that's Not for Me!

This may sound overly cynical, but I have had too much experience with mathematics on too many levels to back off very far from this position. One of my most bitter mathematical memories has to do with the charmingly titled little book *Three Pearls of Number Theory,* by the famous Russian mathematician A. Y. Khinchin. This book, written for the admirable purpose of helping a wounded soldier-friend of Khinchin's pass several dreary months in a hospital, contains a proof by Khinchin of an absolutely beautiful result in number theory called "Van der Waerden's theorem". We can skip the theorem entirely; what matters is simply that Khinchin's proof is very dense in symbolism and extraordinarily hard to follow. I nevertheless loved the theorem so much that I simply *had* to plow through its proof, no matter how hard. I read it through twice, each time taking me several hours. At that point, I had finally digested and internalized it fully, and I realized that it was really very simple. The frightening thickets of double-subscripts turned out to symbolize very *visual* ideas! I could see it all in pictures in my mind.

By spending another few hours, I managed to "translate" Khinchin's proof into a set of a dozen or so elegant colored diagrams. With the aid of these pictures, I was able to take an intelligent nonmathematician friend of mine — someone, in fact, who claimed to dislike math — through the entire proof in every last detail in under half an hour. My nonmathematical friend seemed to get the idea rather easily, whereas it had taken me many hours of struggle to get it.

The way I think about this is that my set of pictures, not Khinchin's obscure equations, was the *real* proof — that is, the set of ideas that Khinchin himself had in mind — and that what Khinchin published was *deliberate obscurantism.* If it wasn't that, then it was at least *intellectual spartanism* of an extreme sort. Why in the world did he show no pictures? Why did he make it so hard when in reality it could be made so intuitive? Why did he have the gall to imply that his book was a collection of precious pearls when it was so needlessly obscure? I think he would have done better to call his book *Three Oysters of Number Theory*.

Countless experiences like this have made me very cynical about dense symbolism accompanied by a total absence of pictures. I realize that one cannot generalize to all situations, but there is certainly *something* to my impression that much of mathematics is made to seem much harder and more complex and more profound than it really is. There is often a very simple and crystal-clear diagram hiding in a dense thicket of symbolism.

But let me return to my visit to the math library. I certainly came up empty-handed in terms of looking for previously published statements of my discoveries. I also came away extremely depressed. Out of the roughly 100 articles that I had surveyed in the *Journal of Geometry,* exactly *one* belonged to the domain of "elementary Euclidean geometry". Most of the rest were concerned with generalizing generalizations that had already been generalized once or twice before. It reminded me of a cynical remark I'd once made about mathematicians:

> Definition of a mathematician: Someone who, on first learning about sex, says, "Just two? That's the trivial case... Let us consider the case of continuum many different sexes."

I am not a sworn enemy of generalization, by the way. I recognize its appeal and its beauty. I have even engaged in it myself! However, there is a kind of paradox associated with generalization that I do not fully fathom, and that is the fact that whereas the act of generalization is supposed to free one up from specifics and to carry one into, well, very *general* realms, what more often seems to happen is that generalization leads down increasingly narrow alleys, so that in the end the papers involving the highest levels of generalization are comprehensible only to a tiny group of people who enjoy counting angels on the head of a pin. In other words, generalization somehow usually leads to trivia. As I say, I don't fully understand how or why generalization is so often self-defeating, but there seems to be some kind of trick whereby one can walk a fine line between concreteness and abstraction, between specialization and generalization, in such a way that the results are deep, powerful, and comprehensible — but only a few people seem able to pull this trick off. Most people who stay in mathematics succumb to the "pleasures" of n^{th}-order generalization, where n goes to infinity.

I cannot really judge the articles in the *Journal of Geometry.* My intuition tells me that many of them must be shallow despite their air of depth, but surely some of them *are* genuinely deep and important. Sometimes I feel positively daunted by the remoteness and incomprehensibility of the whole journal, and I feel a kind of childish admiration for anyone who can think at such abstract levels. But I oscillate between respect and disgust. It is a very strange and uncomfortable feeling.

After I came home from the math library, the enjoyable if ridiculous fantasy from a couple of nights earlier that my discovery might merit a Fields Medal was replaced by its exact opposite: that this was the most elementary, trivial, childish, outmoded type of mathematics imaginable, at no higher than a high-school level. I found myself going through mental flip-flops that left me utterly baffled — nay, *bewildered.* When I thought about my result on its own, just for what it was, it seemed to me I had found something very elegant, something that made even the stunning Euler line seem like just a *hint* at what was really there to be found. It felt like a real advance! But when I then imagined those endless shelves of books and journals filled with incomprehensibly abstract results, making my kinds of ideas look positively infantile, I felt nothing but shame and insignificance.

At about this point, I remembered a passage I had read only a few days earlier in the highly acclaimed book *The Mathematical Experience*, by mathematicians Philip J. Davis and Reuben Hersh.

> In mathematics, many areas show signs of internal exhaustion — for example, the elementary geometry of the circle and the triangle, or the classical theory of functions of a complex variable. While one can call on the former to provide five-finger exercises for beginners and the latter for application to other areas, it seems unlikely that either will ever again produce anything that is both new and startling within its bounded confines.

Even before making any geometric discovery of my own, I had felt a sense of outrage upon reading this quote. The tone was both insulting and narrow-minded. And now, of course, I felt that way all the more strongly. Five-finger exercises, indeed!

It is instructive to contrast this pessimistic outlook with Coolidge's assessment of the future of the field in 1916. Here is the paragraph that opens the concluding page of his Chapter One, the quasi-book called "The Circle in Elementary Plane Geometry":

> It is a parlous undertaking to suggest possible lines of further advance in the subject of plane geometry. On the one hand, the subject has shown itself inexhaustibly fertile, new discoveries have come in such numbers at times when a superficial observer would have felt sure that the last word had been said, that it would be highly unwise to assert that with a little patience one might not strike oil by working in any portion of the subject. On the other hand, the existing literature is so vast that there is a large antecedent probability that any new seeming result may have been discovered decades if not centuries before.

Now this charmingly written little statement, in contrast to Davis and Hersh's condescending pontifications, rang true to my ears. And interestingly enough, I found a similar summary and outlook for the future at the end of virtually every one of Coolidge's 15 wonderfully scholarly and deep chapters! Words like "limitless" "inexhaustible", and "illimitable", as well as phrases such as "much remains to be done" and "there must be a large amount of treasure to be unearthed", kept on cropping up in these summaries. In fact, it is worthwhile quoting how Coolidge brings his book to a close, because it is representative of his entire style and attitude.

> What is certain is that the circle has been diligently studied for two thousand years, and that it will be similarly studied for many thousands more. The methods of attack here exhibited are no more in advance of those known to Euclid and Apollonius than will be those of future geometers in comparison with the best that we have been able to show. This, at least, is what we have a right to hope and expect. For ourselves, 'Let us shut up the box and the puppets, for our play is played out.'

Certainly Coolidge in 1916 expected that geometry had a rich future, even without its *methods* changing to any large degree, ahead of it. And I doubt that much has occurred in the intervening decades to change that.

It so happened that a couple of weeks earlier, I had heard for the first time a tape recording of my late father, Robert Hofstadter, reminiscing during the last year of his life with his old friend Robert Herman about their lives as physicists, and their feelings about their discoveries and their colleagues and various other topics. The same day as I had come across the upsetting quote in Davis and Hersh's book, I had received that cassette in the mail from Herman, and quite by coincidence, listening to it for the first time that evening served as a superb antidote to Davis and Hersh.

In one part of their conversation, my father recalled that in 1946, he had been given the assignment of teaching optics to a small class of physics graduate students at Princeton. The students, knowing that optics was an ancient and very classical topic, were uniformly skeptical that this field could have any bearing on their future research careers. They kept on saying, "Isn't optics a dead field, a closed book?" But my Dad argued with them that such an appearance was certainly deceptive. He expressed an unwavering faith that the classical field of optics was *too beautiful to be exhaustible.* This may not have held any water with the students, but it was his honest opinion. And then my Dad and Herman chuckled together, saying that neither of them could possibly have anticipated in 1946 just *how* optics might progress, but that in fact the unpredictable discovery of the laser in the 1950's and 1960's had totally revolutionized the field, making it one of the hottest and most central fields of modern physics. This uplifting idea that classical beauty and simplicity are precisely what *prevent* a field from being exhaustible seemed quite contrary to the spirit of the remark by Davis and Hersh, and resonated exactly with my sentiments. (Of course, that is not too surprising, since my Dad's ideas had seeped into my very soul ever since I was a tiny child!)

Might plane geometry be comparable to optics, or is this mapping of two "classical" fields a false analogy? Someone might argue that the laser brought "foreign" ideas — ideas from quantum mechanics — into optics. This would suggest that the apparent fertility of optics didn't really stem from some *internal* aspect of the field, but from a transplant. According to this viewpoint, then, for plane geometry to be renewed, ideas from *outside geometry* would have to be brought in. Davis and Hersh seem to say as much when they write, "it seems unlikely that either will ever again produce anything that is both new and startling *within its bounded confines.*" Perhaps they would agree that an influx of outside ideas, coming from some other branch of mathematics, might revitalize the geometry of the circle and triangle.

But this viewpoint seems quite opposed to Coolidge's closing words to the effect that Euclid's methods were good enough for him, and ought to be good enough even for future generations! Of course, his claim was a bit exaggerated: in truth, there have been many great advances in technique since Euclid, and in fact they permeate Coolidge's book — but in a certain sense, geometry has all the while remained *accessible* and *concrete,* and that is probably what Coolidge really meant. Geometry has remained what I call a "horsies-and-doggies" subject — one that doesn't take a novice too long to get into, one that a good high-school or college student can play around in and perhaps make genuine discoveries in.

Or rather, up until Coolidge's time, geometry *had* remained accessible. Today, as my selection of titles and theorems from the *Journal of Geometry* shows, what is called "geometry" seems arcane and forbidding, even to someone who was once deeply in love with mathematics. Today, it almost seems that there is no place to publish ideas like the ones in this essay, except possibly in unprestigious "teaching journals". It's not considered *research.* One gets the distinct impression that *real* mathematicians wouldn't be caught dead talking about *triangles*! And that seems sad.

A couple of months ago, I was in Italy and had the chance to see my good friend Benedetto Scimemi, a mathematician at the University of Padova, for the first time in a couple of years. Nonchalantly, I remarked to him that of all things, I had of late become a maniac about elementary geometry. To my amazement, Benedetto replied, "So have I." "Really?", said I, and added, "I'm in love with the special points of a triangle." Benedetto said, "So am I." Then I said, "What set me on fire was reading Coxeter and Greitzer's *Geometry Revisited.*" Benedetto said, "Same here." This was getting strange. Finally I said, "I'm using a computer program to explore these ideas." Benedetto replied, "Geometer's Sketchpad, naturally. So am I." I was floored!

It turned out Benedetto and I really were looking at and eating up the same kinds of things, but of course we had differing approaches. This was a delightful surprise! But Benedetto has also remarked more recently, via electronic mail, "My colleagues are simply turned off by this Euclidean-geometry stuff. It's perceived as completely out of fashion, and you can't publish it in any 'important' journals. If you tell your colleagues about your interest in this kind of thing, you run the risk of evoking condescending smiles." I find this *truly* sad.

These days, there are a couple of places where what I think of as beautiful mathematics is discussed clearly and in an exciting way. The *Journal of Recreational Mathematics* has lots of pictures and lots of genuinely exciting ideas. So does *Mathematics Magazine,* a journal intended for undergraduates and math teachers. (For that matter, so did Martin Gardner's venerable "Mathematical Games" column in *Scientific American,* which purported to be merely about "games", whereas in fact it exemplified play with ideas at the highest level of beauty and importance.) But if you tried to survive as a professional mathematician by publishing in the *Journal of Recreational Mathematics* or *Mathematics Magazine,* you would become an object of many snorts, and would have no hope of getting tenure at a top-rank university.

It would be nice if there were a journal that was taken seriously and yet where ideas like the hemiolic crystal were welcome. I don't suppose the title *Journal of Horsies-and-Doggies Mathematics* would go over very big, however.

Sometimes I feel like a stranger from another era, both mathematically and musically. I love songs from the thirties and thereabouts, but I know that if "A Fine Romance" or "Not for Me" were played on a pop-music radio station today, it wouldn't matter one bit how melodious, witty, or sparkling the song was — listeners would be completely turned off. In fact, I suspect that precisely those qualities would turn them off! There's such a stylistic gulf that pop-music listeners today cannot relate to what I consider beauty and charm. But this cultural tendency seems arbitrary rather than inevitable. I don't think that heavy metal represents musical *progress* over the style of Gershwin and Kern and Rodgers and Berlin and Porter. Nor do I think that disdain for the concrete represents mathematical progress over the style of Brocard and Crelle and Möbius and Steiner and Coolidge.

Perhaps the musical analogy most pertinent to this discussion concerns the claims made every so often to the effect that tonal music — the language of Bach, Bizet, Bechet, and the Beatles — is an exhausted medium. Pushing a kind of dogmatic avant-gardism, some musical commentators proclaim, "There is nothing fresh left to say in the tonal idiom." This strikes me as about as likely as the idea that coherent, grammatical prose is a dead language and that no high-quality novel or short story will ever again be written in that medium. People who make such bald claims do nothing but reveal the poverty of their imaginations. Is Davis and Hersh's claim about classical geometry any more plausible?

What if a physicist made the same claim of exhaustion about classical mechanics (which had its heyday in the eighteenth and nineteenth centuries, but was supplanted by quantum mechanics in the first quarter of this century)? One argument seemingly in favor of such a claim would be that *classical mechanics was shown to be wrong.* So how could it be at all meaningful to work in that field any more? The flaw in this silly argument is that classical mechanics is an internally consistent system as well as a canonical limiting case of quantum mechanics; in fact, one can't hope to understand quantum mechanics without first having absorbed and mastered classical mechanics. This is because humans naturally think in classical terms. So classical mechanics has hardly been jettisoned; it plays a central role in physics, and always will.

A parallel debate could be conducted concerning geometry. Argument: Doesn't general relativity show, with its curved spacetime, that our universe is not Euclidean but non-Euclidean — and thus that old-fashioned *Euclidean geometry is wrong*? Like the argument above, this is silly. Euclidean geometry is an internally consistent system as well as a canonical limiting case of non-Euclidean geometry; in fact, one can't hope to understand non-Euclidean geometry without first having absorbed and mastered Euclidean geometry. This is because humans naturally think in Euclidean terms. Therefore Euclidean geometry, whether applicable or not to our physical universe, plays a central role in mathematics, and always will.

The real question about both classical mechanics and Euclidean geometry is: Are they still fertile areas for investigation, or have they both come to an end, for all practical purposes? Is it just fashion and caprice to consider them no longer worthy of serious research? I think it is just fashion and caprice, but I don't know for sure. Could some physicist, one fine day in the future, be awarded the Nobel Prize for making novel and elegant advances in the venerable old field of classical mechanics? I admit that this sounds somewhat implausible to me, but I can't put my finger on just why. Maybe it's simply that I have internalized the viewpoint that today's physics culture has collectively established. And, some fine day in the future, could a major advance in Euclidean geometry come to be widely considered as... well, as a *major advance*? I would like to think so, but I am somewhat dubious about this as well.

This idea of exploration in Euclidean geometry gradually slipping into the status of a trivial activity was reinforced most disturbingly for me when I finally got a hold of an article that Scimemi had described to me when we saw each other in Italy. This was David Gale's "Mathematical Entertainments" column in the Spring 1992 issue of the *Mathematical Intelligencer*. Reading it provoked considerable inner turmoil in me.

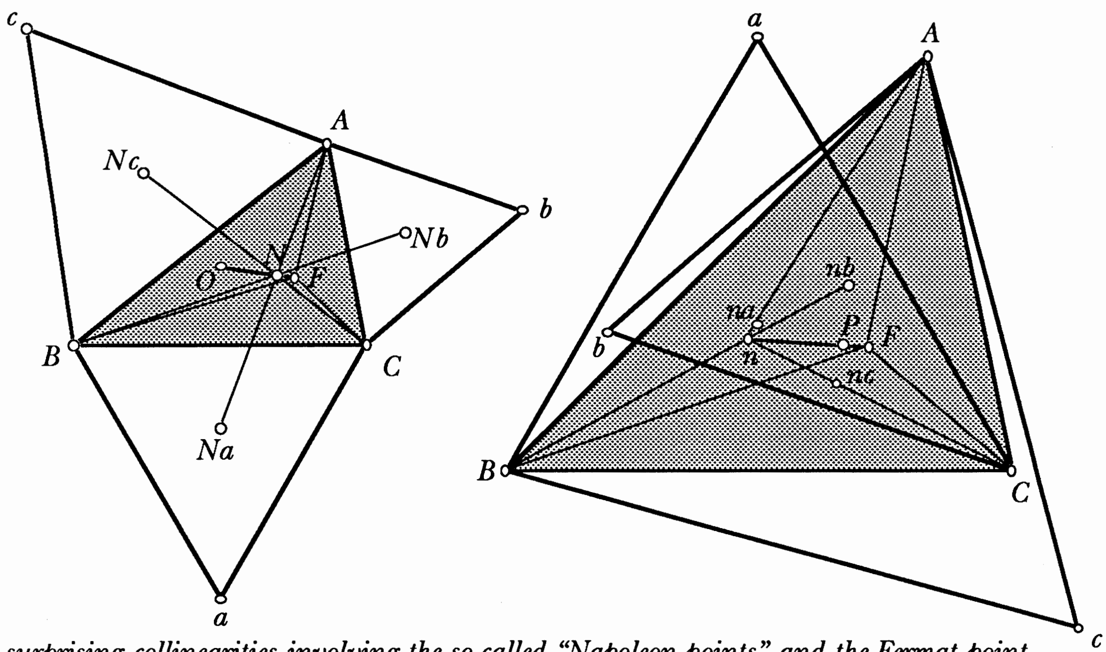
In his column, Gale told of recent work done in geometry by a mathematician named Clark Kimberling. Like me, Kimberling is interested in the special points of a triangle, and like me, he explores their interrelationships with the aid of a computer. The difference is, he does it in an *entirely* automated way. It's a very big difference.

The goal of the study described by Gale was to discover *collinearities* of special points, or "centers", as Gale calls them — apparent "coincidences", like the Euler and Nagel segments, that are really not coincidences at all, but mathematical necessities: hidden consequences of the definition of a triangle. To do this, Kimberling fed into a program the algebraic or trigonometric characterizations of 91 different "centers" — more than I know of, by a long shot! — and had the computer calculate their coordinates, in several randomly-chosen triangles, to roughly ten decimal places each. Then the lines they formed could be generated and numerically compared. Any ten-decimal agreement noticed in one triangle and echoed in other triangles could not be a coincidence, but had to be a *co-incidence,* a shoo-in for theoremhood.

Of course any *two* points determine a line; the question was simply how many *distinct* such lines there were. If there were no collinearities at all, as would be the case if the points were randomly chosen and thus completely unrelated to one another, there would be 4095 different lines — one for each pair. But these "centers" were not random points by any means, and numerical comparison (by the machine, of course) revealed an enormous degree of order: all 91 centers lay on a mere 103 distinct lines — "special lines", we could call them.

In my view, this discovery is a completely new kind of advance in Euclidean geometry. It is a *tour de force* that couldn't have been imagined before our times, and it gives us a qualitatively new type of insight into what goes on in triangles. Which points have the most lines going through them? Which lines have the most points? Could one use the intertwined point/line incidence statistics to *objectively rank-order, by their importance, all the special points and lines*? This is a strange but fascinating idea.

One of the individual results Gale cites is the discovery by Kimberling's program that, in addition to the standard *four* centers *OGPH* on the Euler line (and now I really do mean *line,* since we are talking about the extended line that goes beyond *H* and *O*), there are *eight more centers* on that line! (He doesn't tell us anything about them, unfortunately.) Gale also cites an elegant pair of closely-related theorems, shown below, that were discovered by Kimberling's program.



Two surprising collinearities involving the so-called "Napoleon points" and the Fermat point.

On the left, collinearity of N, F, and O. Here, N is ABC's "outer Napoleon point", which is where the lines joining the vertices of ABC with the centers Na, Nb, Nc of the three external equilateral triangles all meet. F is the Fermat point, where the lines from ABC's three vertices meet each other at 120° angles. O is ABC's circumcenter.

On the right, collinearity of n, F, and P. Here, n is ABC's "inner Napoleon point", which is where the lines joining the vertices of ABC with the centers na, nb, nc of the three internal equilateral triangles all meet. F is the Fermat point, defined as above. P is the center of ABC's nine-point circle.

These two collinearities were first discovered by a computer program written by Clark Kimberling. Both facts were also first proven by another computer program by Kimberling.

Each unexpected collinearity — and there are a great number of them — is a theorem, or at least a strongly supported conjecture needing proof. And here is where Kimberling makes ingenious use of the computer again. The algebraic or trigonometric characterizations of the various centers are symbolic expressions that can be manipulated in a tireless way by any sophisticated "computer algebra" program, such as *Mathematica.* Proving that three such centers are indeed collinear is, in principle at least, a mere exercise in symbol manipulation — a task at which computers excel. Realizing this, Kimberling simply handed over to *Mathematica* the chore of *proving* all of his computer-generated conjectures, which it compliantly did for him (including the Fermat–Napoleon pair shown above). Let me now quote from Gale's column, because Gale puts this all in perspective about as well as it can be:

> Surely this is a rather strange state of affairs. Everything is being done by the computer. Program A goes on a voyage of exploration and comes up with a vast number of theorems. Then program B takes over and supplies the proofs, and while all this is going on the investigator just sits back and watches. The robots have taken over. It makes one reflect a bit on what we are trying to achieve in doing mathematics. It is certainly impressive to suddenly learn hundreds of new facts in a discipline that people have worked in for more than two millennia. But mathematics, and science generally, is concerned with much more than compiling a huge catalogue of facts. The hope is to find general principles from which the facts can be deduced, and the robots don't seem to be very helpful for this. They tell us what is true but don't tell us why. They supply lots of information but little insight.

Is this not all rather remarkable? What would Euclid have thought? What would Euler or Poncelet or Coolidge have thought? What do *we* think?

The first time I heard about Kimberling's work was Benedetto Scimemi's capsule description, and I was very upset. After all, I was in love with the special points and their interrelationships, and had hoped to be a big discoverer of scads of new theorems about them. Now, it seemed that I was not only going to get scooped, but scooped by a machine! (Of course, I was sneakily planning to use a machine myself, but I certainly wasn't going to say that the machine had done the research!)

Learning more from Gale's column about Kimberling's successes did not cheer me up. In fact, since I read the column just a couple of days after making my big discovery, I immediately started worrying that the crystal — *my* crystal! — might be among the scads of results churned out by his machine. What a strange coincidence it would be for a human and a machine to have independently and contemporaneously come up with the same fundamental new finding about triangles after a 60-year hiatus in the field!

Even though I hated the idea that my discovery might not be mine because I came along just a few months too late, I nonetheless found the whole Kimberling story extremely thought-provoking, and wrote to Gale to find out Kimberling's address. In my letter, I reflected as follows on my perplexity:

> I find the interconnections of these "centers" of a triangle beautiful and mysterious. Each such relationship might be likened to a little gem of a poem. Therefore I was in a way upset to hear that these little "poems" are being turned into trivia — at least in the sense of new ones now being producible in massive quantities in a mindless, mechanical manner. We would certainly not like it if we found that a computer could be easily programmed to come up with deeply moving poems in a mindless manner. It would greatly reduce our feeling that poems written by humans were moving or charming if the same effect could be produced without any sentiment or insight behind it.
>
> On the other hand, despite being somewhat saddened, I was also fascinated, because these relationships among special points or "centers" have for me an indescribable quality of depth and mystery to them. The more of them I know, the better. Who cares whether they were produced by a human, a machine, or an oracle? I feel that each new one brings me ever so slightly closer to fathoming the mysterious "essence of triangularity", even though I will certainly never attain that ultimate "nirvanic" state of understanding.

If a computer had done in, say, 1955 what Gale describes in his column, it would have been considered earthshaking news, and the popular press would doubtless have touted it as an irrefutable demonstration that intelligent computers existed, even that *computers can think.* After all, what higher mental activity than creative mathematics? What more sublime accomplishment than the discovery of brilliant new diamonds in Euclidean geometry? That's how it would have been seen in the fifties, perhaps even by sophisticated observers. But for us in the 1990's, is there any reason to see it differently? I think so.

A very large monetary prize was put up, some years ago, for "the first significant new mathematical theorem discovered and proven by a computer". It has not yet been given out. Could either of Kimberling's results concerning the Fermat and Napoleon points conceivably merit this prize? They are almost certainly new. They are indisputably elegant theorems. They were unquestionably discovered and proven by a computer. So only one question would seem to remain: *Are they significant?*

The level of significance of new results in Euclidean geometry is of course debatable, as my pained musings about the worth of my own discovery clearly show. But it should not surprise anyone to hear that I lean toward the opinion that these two new theorems found and proven by Kimberling's machine are worthy findings. If a human had found them, the achievement would perhaps not be considered deserving of a Ph.D. in mathematics, but I would guess that a somewhat larger selection of the machine's results — 10 or 20 theorems, say — would be every bit as novel and as interesting as the vast majority of Ph.D. theses in mathematics. Or just for fun, let us assume, as in my nightmare, that Kimberling's machine found my crystal, my very own crystal, and went on to discover many more of its properties than I have. (And for all I know, it actually did so!) Wouldn't *that* be grounds for awarding the prize to the machine? Or to Kimberling?

Ah, but there is the crux of the matter. Which of the two deserves the prize? The program certainly wouldn't appreciate receiving an award, because the program is insentient — in fact, totally dumb. No matter how much one admires its output, one *still* feels that there was one and only one mind doing any mathematics of any sort here — and it was Kimberling's. Kimberling, not the machine, was the one who wanted to know how special points interrelate. Kimberling, not the machine, was the one who thought up the technique of numerical search using random triangles, and the exploitation of a computer-algebra program. Kimberling, not the machine, selected 91 special points to be fed in, and Kimberling, not the machine, knew their trigonometric characterizations. When you subtract all this out, you are left with an unconscious, brute-force search not all that different in flavor from the brute-force calculation of a billion digits of π, which of course nobody considers a mark of genuine intelligence, let alone a creative or significant mathematical act. (Of course, maybe a *trillion* would be creative...)

The most charitable assessment (too charitable, but leave that aside) would be to call this research *joint work*. For example, probably most of the results generated by the machine had considerably less esthetic appeal than the two Gale cited. I wouldn't be surprised, therefore, if Kimberling had to wade through a good many boring results to find a truly beautiful one. It would of course be far more impressive if the machine itself had a sense of esthetics and surprise, and once in a while printed out, "Hey, Clark — what do you think of *this* one? Isn't it a beaut?" But that wasn't what Kimberling was trying to do. He wasn't trying to automate the *doing of mathematics* — he was trying to automate the *discovery of fresh new theorems of geometry*. There is a world of difference, when you think about it carefully.

So I was being disingenuous above when I wrote that these theorems were "unquestionably discovered and proven by a computer", and that the only remaining issue was that of their significance. Unquestionably, my foot! Hidden in phrases like "discovered by a computer" — phrases that apply mental terms to computers — lurk some tacit assumptions about *computers as autonomous agents* versus *computers as tools,* which we have now brought out into the open. A discovery made *by* a computer and a discovery made *through* a computer are two different things indeed. We don't say that telescopes make discoveries in astronomy, even if they are computer-controlled!

The question of whether to award the prize for Kimberling's work in geometry seems to me to depend crucially on whether the prize was intended to celebrate the emergence of the era of automous computer creativity, or just to give credit to people who creatively use a computer as a tool (and certainly Kimberling did that, in spades). I'm quite sure that the former was the intent, and therefore, if I were a member of the prize committee, I would be against awarding the prize to Kimberling and/or his machine, no matter what level of significance is attributed to their work by the mathematics world.

Forty years ago, people were much less sophisticated at thinking about the distinction between computers as unjudgmental generators of information and computers as reflective, autonomous agents. The human mind itself was much less clearly understood, and the extremely different strengths and weaknesses of minds and machines were still to be revealed by the various successes and failures of artificial intelligence. Forty years ago, finding and proving beautiful and completely new theorems in geometry, no matter how it was done, would have been considered astonishing. Even a reasonably competent "computer algebra" program (and good ones are almost a dime a dozen today) would have been considered a major step towards the mechanization of the highest levels of mentality!

But today, we don't jump to such conclusions. If Kimberling's machine by luck discovered the hemiolic crystal, we would easily recognize the huge difference between its act of discovery and mine. In one case, the discovery would be motivated by deep *curiosity* and would involve such quintessential cognitive processes as the manipulation of *concepts,* the making of *analogies,* the ranking of ideas as to *importance,* the recognition of *elegance,* and the feeling of *surprise.* In the other case, the discovery would be made in an indiscriminate, brute-force manner with no making of analogies, no judgments of importance, no genuine concepts motivating it — not even basic concepts like "triangle" or "line" or "point", let alone meta-level concepts like "interesting" or "surprising" or "elegant". In one case, it would be fair to say that the discovery was self-motivated, whereas in the other, one would have to concede that no decision whatsoever came from within the "deciding agent" itself. In short, the computer as used by Kimberling epitomizes Lady Ada Lovelace's famous remark, made in her 1842 memoir about Babbage's Analytical Engine, to the effect that a calculating machine *does precisely what it was told to do, and no more.* (Her actual words were these: "The Analytical Engine has no pretensions whatever to *originate* anything. It can do whatever we *know how to order it* to perform.")

Could any computer *ever* escape the Lovelacian epithet? Could a computer ever attain true autonomy? This is in fact my research area, and I have thought about it for many years. I firmly believe the answer is "yes" — and I think it will happen when computers, like people, have *flexible concepts* whose associative halos shift according to situation; when computers, like people, constantly make judgments about *what matters* and what doesn't, about *what is interesting* and what isn't, and about *what is surprising* and what isn't; when computers, like people, can be *reminded* by a new situation of something that happened before; when computers, like people, are *aware of their own processing* at some coarse-grained level, and use that knowledge to guide themselves; when computers, like people, *make mistakes* and recognize them and learn from them; when computers, like people, have a sense of *beauty and curiosity* and are driven by it. All this is not around the corner, to be sure, but programs whose architecture has many of these features to a small degree are beginning to be designed. I don't think we're likely to see genuine machine creativity — creativity where the credit clearly should redound to the machine and not to its developers or "coaches" — for decades if not centuries, but at least we are gradually developing a sophisticated understanding of what it means to *talk* about such things.

But notice a terrible irony here: The more one thinks Kimberling's geometry program is a mindless automaton, the stronger the argument that it has thoroughly trivialized the further exploration of Euclidean geometry, rendering the subject a mere historical curiosity, totally irrelevant to modern mathematical research. Or so it would appear on first sight. Actually, I think matters are more complex than this, and so, it would seem, does David Gale. He clearly felt that something crucial was lacking in what Kimberling's "robots" do. As Gale put it, "They tell us what is true but don't tell us why". I think this critique comes close to the mark, but misses it slightly. In my letter to Gale, I tried to articulate my view:

> The first program discovered, via numerical experimentation, a lot of new facts. The second program discovered the *proofs* of those facts. Now traditionally in mathematics, people have not distinguished between *proving* a fact and giving the *reason* for that fact's truth. In other words, *proving* is universally assumed to be the same as understanding *why.*
>
> But everyone knows that some proofs are opaque while others create clarity. This measure of transparency is, I think, what you were implicitly referring to when you said that the computer proofs don't tell us *why* the statements are true. But the English word "why" doesn't quite say enough. I think what you meant would be better expressed as follows: "The robots tell us *what* is true, and in a sense they even tell us *why* it is true, but

they don't furnish us with *whys of insight.*" The fact is, there are "opaque why's" and "clear why's" (and of course all possible shades in between), and the computer — at least in this case — furnishes only whys that lie toward the opaque end of the spectrum.

At the end of his column, Gale wrote in favor of intuitive understanding and against rigor for rigor's sake. I couldn't agree more. I have always felt that mathematicians give too much weight to proofs and not enough to clarity. So this brings up the following question: Is the only way one can come to deeply *understand* a mathematical truth via a proof? Or are there other ways of deep understanding that fall short of being proofs? After all, we surely understand many phenomena in the real world as deeply as we could ever understand *anything* in mathematics, yet we do so entirely without proof, merely through experience. Math is the only field where we insist that proofs and understanding are synonymous. But is this justified? Why couldn't a *why of insight* come in math, as in the real world, from experience?

Could watching computer-graphics displays of mathematical phenomena, for instance, provide an alternative route to deep understanding? Couldn't our ability to pick up complex patterns through vision substitute, in some situations, for the intellectual exercise of a rigorous demonstration? Or alternatively, could one come to understand some mathematical fact or phenomenon with great clarity simply because it is *analogous* to something else with which one is very familiar? I suspect that an uncanny and mostly unconscious facility with analogies is what gave Ramanujan his unerring way of arriving at new mathematical truths — and it so often bypassed proof entirely, confounding his more orthodox colleagues.

Questions like this probe long-standing assumptions at the very heart of mathematics. For this reason, I find Kimberling's computational studies to be every bit as philosophically provocative as the now-infamous Appel–Haken proof-by-computer of the four-color theorem. They make one wonder, *What is math all about? What are creativity and discovery all about?*

One thing that seems clear to me is that indiscriminate generation of information, even when it includes marvelous gems, does not constitute doing math or science. Again, Gale comes close to making what I think is the right point when he says, "Mathematics, and science generally, is concerned with much more than compiling a huge catalogue of facts. The hope is to find *general principles* from which the facts can be deduced." True, but a bit misleading. Even general principles are of no value if they are generated but then go unrecognized because of being randomly scattered among huge piles of less important facts.

George Gamow in his delightful book *One, Two, Three... Infinity* and Jorge Luis Borges in his delightful short story "The Library of Babel" both describe huge libraries containing every possible book, composed by the mindless mechanical act of combining symbols into strings. Both authors vividly convey the hopelessness of finding *useful* items in such a library, filled as it would be with such useless passages as "aaaaaaaaaaa...", "boobooboobooboo..." (both patterns repeated forever), "zawkporpkossscilm..." (blathering on randomly without any part ever being recognizable in any language at all), "horse has six legs and..." (an endless recitation of falsities in good English), "I like apples cooked in terpentin..." (an endless litany of incoherencies riddled with typos, for good measure). It's quite obvious why this would be such a useless place to visit.

But now imagine a variation on this theme: The Library of All Mathematical Truths. On the surface, it sounds infinitely better than the Library of Babel. After all, included in this library would be all truths of arithmetic (1+1=2, 12x12=144, etc.), all the gems of Euclidean geometry, all the beautiful theorems of number theory, all of real and complex analysis, all of topology, group theory, and category theory, all the recent foundational work in set theory and metamathematics, even Gödel's incompleteness theorems and their endless spinoffs... Every deep idea ever proposed or that ever *might* be proposed would be found here, as long as it is true. So every last one of Gale's vaunted "general principles from which the facts can be deduced" would be here. Sounds great, right? The problem is that there would be unimaginable amounts of useless garbage as well — true but useless trivia — completely burying the general principles, rendering them utterly unfindable and thus utterly useless. In fact, this starts to sound quite a bit like your typical math library, come to think of it.

Doing mathematics has only a vague relation to knowing lots of facts or even being a great technician in some specialty. Drones can attain such competencies, sad to say. Often, these abilities only becloud truth; what really matters is *knowing where to put the emphasis.* In other words, what matters in mathematics (and in science in general) is the ability to reliably distinguish the very important from the merely interesting, the merely interesting from the mundane, and of course the mundane from the trivial. Without this, one is not doing mathematics in any genuine sense. One might as well be a Kimberling robot.

And let's not knock the Kimberling math robots too much, anyway. What they do comes pretty darn close to what mathematicians often *say* doing mathematics is all about: finding new truths and proving them. Of course, people who say this usually haven't thought about it too much. Math really *is* about Gale's "general principles" — but very often those principles go unstated and unrecognized. Why in the world is that so? The answer is simple: the general principles that underlie all serious progress in mathematics are not theorems but *intuitions* and *images.* Moreover, such wispy mental phenomena are most often born out of *analogies.* Two theorems seem reminiscent of each other in some abstract way, and a vague image crystallizes in someone's mind. All of a sudden, a torrent of new theorems pours out! But in the articles that ensue, the analogy that sparked it all may never be mentioned at all. How come? Because it is not in itself a *formal notion,* and mathematicians have had it drilled into their heads all their lives that *definitions, theorems,* and *proofs* are what math is about. Of course, those are a big part of math, but there's something beyond them.

George Pólya made a laudable effort to put his finger on these ethereal kinds of patterns in mathematical thought in his two scholarly volumes *Induction and Analogy in Mathematics* and *Patterns of Plausible Inference,* as well as his more popular book *How to Solve It.* Unfortunately, Pólya was exceptional. Few mathematicians have the courage to write about their images and intuitions — after all, what could be worse for a mathematician than being caught saying something *wrong* or *vague*? Perhaps this is why Khinchin didn't draw any pictures, even though he surely had pictures — my pictures — in his mind. Instead, he churned out a raft of formidable formalistic stuff, because doing so was very orthodox, and therefore very *safe.* Maybe I'm wrong about Khinchin, but I suspect that this is true of a large percentage of mathematicians. Professional insecurity, then, makes mathematicians do a pretty good job of imitating Kimberling's mindless math robots!

Here is an amusing analogy. Maybe each individual human mathematician is like a mindless Kimberling robot spewing large numbers of raw mathematical truths (*i.e.,* articles) into the Library of All Mathematical Truths, and the mathematical community as a whole is a bit like Kimberling, sifting and weighing the articles, judging their levels of interest and importance. In this analogy, the highest intelligence — in fact, the *only* intelligence! — would reside in the math community as a whole. Of course this is a joke, but is there not a grain of truth in it anyway?

Perhaps the best characterization I have found of mathematics is this: "Math is the study of the beauty of the interrelationships of patterns", with strong stress on the word "beauty". The indispensable role of esthetics is the crux of the matter, and it's what makes math-making so extraordinarily hard to mechanize, to model.

Math-making being as elusive as it is, I feel it could be instructive to make a few high-level comments about this particular act of geometry-making, an act that I was fortunate enough to be able to watch from a ringside seat as an ongoing process in my own mind. First of all, as I hope to have made clear, I am not much of a geometer. I am just a geometry amateur, and something of a bumbling amateur, at that. More often than I like to admit, I have "false epiphanies" in which I mistake trivia for profundities, and I have a rather primitive idea of how to construct proofs. Compared to a titan like H. S. M. Coxeter, I have but a minuscule storehouse of knowledge. So why in the world was I the lucky one to have found this gem? Although some luck was certainly involved, I don't think it was a complete accident. Here is my feeling about it.

First of all, I am apparently just a bit more taken than are most geometers with the mystery of the special points of a triangle. In fact, I doubt that most would use the word "mystery" in this connection. Yet in these points and their beautiful patterns I sense something almost mystical about the "essence of triangularity". To most geometers, it probably doesn't feel quite that compelling. That in itself is symptomatic of a significant attitude-disparity. And my unswerving fascination has led me on, something like an infatuation or even an obsession, to muse for months on end about special points and their interrelationships. I didn't call Chapter O of this essay "Bewitched" for nothing! So that's the first point.

Secondly, I am someone who for years has been in love with analogies, and with the *concept* of analogy itself. Not only do I use analogies in my thinking and my writing all the time, I am also constantly jotting down other people's analogies, wondering how they arise, pondering what makes some of them good and others bad, and musing about what they say about the subcognitive mechanisms of the mind. For almost 15 years, my professional research has been largely about making a computer model of how people make analogies and use them in the creative process. So you can imagine that when I came across the analogy between the Euler and Nagel segments, I was electrified. It was not just a beautiful analogy, but a beautiful analogy smack-dab in the midst of a field that I was intoxicated with already. It was a mind-boggler of an analogy, and to add to that, I was baffled as to why nobody but *nobody* had mentioned it in their books on geometry since Roger Johnson's 1929 treatise. This struck me as so irrational and narrow-minded that it also served as a kind of goad to my curiosity. Was there something I was missing? Was the Nagel segment, despite its seeming fundamentality, just a trivial sidekick to the Euler segment? Was it worthy of nothing but a footnote (if even that) when the Euler segment was treated like a glamorous movie star? I couldn't figure this out, and the image of these two "twin" segments, each one cutting the other in that strange, lopsided way, just etched itself into my brain.

These two factors together added up to something unique, I guess — this special-points-of-a-triangle analogy *par excellence* started swirling around in my head over and over again. I couldn't let it loose, or rather, it wouldn't let me loose. It was the *analogical unity* of the two segments that caught my imagination, much more than either segment on its own. Coolidge's systematic, point-by-point listing of parallels between them certainly was a critical element, and, I have to say, so was the bizarre fact that he himself voiced no curiosity as to *why* there was this deep and beautiful analogy at the heart of triangularity. The fact that this obvious, salient question went completely unasked was almost as much a source of mystery to me as was the analogy itself.

So I see two factors — (1) being a special-points *aficionado,* and (2) being an analogy nut — adding up to a third key factor, which is a kind of mixture of the two: obsession with a fantastic special-points analogy. This was enough to propel me into the discovery. Being a geometry expert was apparently not a prerequisite to this discovery. Being a mere bumbler was good enough! Of course, I was not a typical bumbler — I was a bumbler with one hell of a tool to help compensate for my bumbling — namely, Geometer's Sketchpad.

Thus, one further key factor that mustn't be overlooked is the fortuitous existence and tremendous power of Geometer's Sketchpad. Somehow, this program precisely filled an inner need, a craving, that I had, to be able to *see* my beloved special points doing their intricate, complex dances inside and outside the triangle as it changed. And my own personality welcomed a computer program to explore mathematics, and felt that it afforded visions of *geometric truth,* an attitude that perhaps would be a little bit less accepted by a traditional mathematician. In short, living in the 1990's and having a Macintosh and enjoying computers was also part of it.

I have to admit, there is one last crucial factor that allowed me, of all people, the privilege of making this discovery (if discovery it is): the incredible downplaying and neglect, by several generations of mathematicians the world over, of the "anonymous" segment so much like Euler's. So, to mathematicians everywhere, for not looking in this direction, I hereby express my great debt of gratitude. Thank you!

Putting on my cognitive-scientist's hat now, I would like to point out something that struck me as I reviewed this chronicle of my discovery process — namely, the large number of analogies, good and bad, that figured critically in it. Here, then, is a list of the main analogies that I think served as guiding or misguiding forces, with a brief comment on each one:

(1) My vague, intuitive feeling that special points in a triangle are very much like special constants on the real line, such as e and π. Since I have deeply loved such constants from childhood, this analogy was in part responsible for getting me so hooked on geometry.

(2) Coolidge's systematic mapping between properties of the Euler segment and properties of the Nagel segment. This was a bolt out of the blue.

(3) Seeing the three parallel lines *HN, OI,* and *PS* as a physical instantiation of the Euler/Nagel analogy — in other words, a meta-analogy that maps a visible geometric diagram onto an abstract analogy.

(4) Looking at the crisscrossing-medians diagram that arose in a simple puzzle, and *recognizing in it* the much more profound Euler/Nagel diagram.

(5) Mapping the new point *V* onto the known points *P* and *S* because of their analogous positional roles in their respective segments, and concluding that *V* ought to be the center of some important circle associated with *ABC.*

(6) The scramble-brained analogy that led me so far astray for several hours one day that I eventually felt compelled to return to Coolidge's book for confirmation of my sanity — where I then chanced upon another key analogy — namely...

(7) The idea of constructing *T*'s auxiliary triangle, arrived at by analogy with two other constructions described in Coolidge.

(8) Looking at the behavior of a certain point marked "*O*" in a certain dynamogram, and *seeing it as* Nagel-point-like behavior with respect to a triangle it was inside.

(9) Seeing the three letters "*ONT*" not just as standing for concepts involved in an abstract relationship, but as symbolizing one side of the crystal.

(10) Jumping from a discovery attached to the *NOT* side of the crystal to the idea that maybe two further analogous results would hold, attached to the *HUN* and *TIH* sides of the crystal.

Some of these — especially numbers 3, 4, 8, and 9 — may not strike you exactly as analogies. Why does recognizing Nagel-point-like behavior, to take just one example, constitute an analogy? At this point, I turn for help to someone whose mathematics and whose musings on mathematics I have always greatly admired — Stanislaw Ulam. As Heinz Pagels reports in his book *The Dreams of Reason,* one time Ulam and his mathematician friend Gian-Carlo Rota were having a lively debate about artificial intelligence, a discipline whose approach Ulam thought was simplistic. Being convinced that perception is the key to intelligence, Ulam was trying to explain the subtlety of human perception by showing how subjective it is, how influenced by context. He said to Rota, "When you perceive intelligently, you always perceive a function, never an object in the physical sense. Cameras always register objects, but human perception is always the perception of functional roles. The two processes could not be more different... Your friends in AI are now beginning to trumpet the role of contexts, but they are not practicing their lesson. They still want to build machines that see by imitating cameras, perhaps with some feedback thrown in. Such an approach is bound to fail..."

Rota interjected, "But if what you say is right, what becomes of objectivity, an idea formalized by mathematical logic and the theory of sets?"

Ulam parried: "What makes you so sure that mathematical logic corresponds to the way we think? Logic formalizes only very few of the processes by which we actually think. The time has come to enrich formal logic by adding to it some other fundamental notions. What is it that you see when you see? You see an object *as* a key, a man in a car *as* a passenger, some sheets of paper *as* a book. It is the word 'as' that must be mathematically formalized... Until you do that, you will not get very far with your AI problem."

To Rota's expression of fear that the challenge of formalizing the process of *seeing a given thing **as** another thing* was impossibly hard, Ulam gave the droll reply, "Do not lose your faith. A mighty fortress is our mathematics." I personally don't think that mathematical formalization is the key to making machines that can "see as", but that was Ulam's opinion. In any case, we can take Ulam's key word "as" and see it as an acronym for "abstract seeing". Then Ulam's thesis becomes "AS is the key to AI", a thesis to which I fully subscribe.

You could look forever at a point moving around on a screen but get nowhere in understanding its motion, unless it were to cause some pre-existing concepts in your unconscious mind to bubble up toward conscious recognition. In my case, I watched a point move around and some aspects of its behavior rang a bit of a bell. The more I watched, the louder the bell got. First the bell was just saying *inside the triangle.* Then it was saying *often near an edge.* Then *often near the longest side.* Then *often down near the pointy end of the triangle.* At roughly this moment, the notion of "Nagel point" burst into my conscious mind, and I suddenly saw the point's motion in a completely new way. I had wheeled in a whole new cluster of concepts in terms of which to frame what was objectively there for cameras to see in their context-free way.

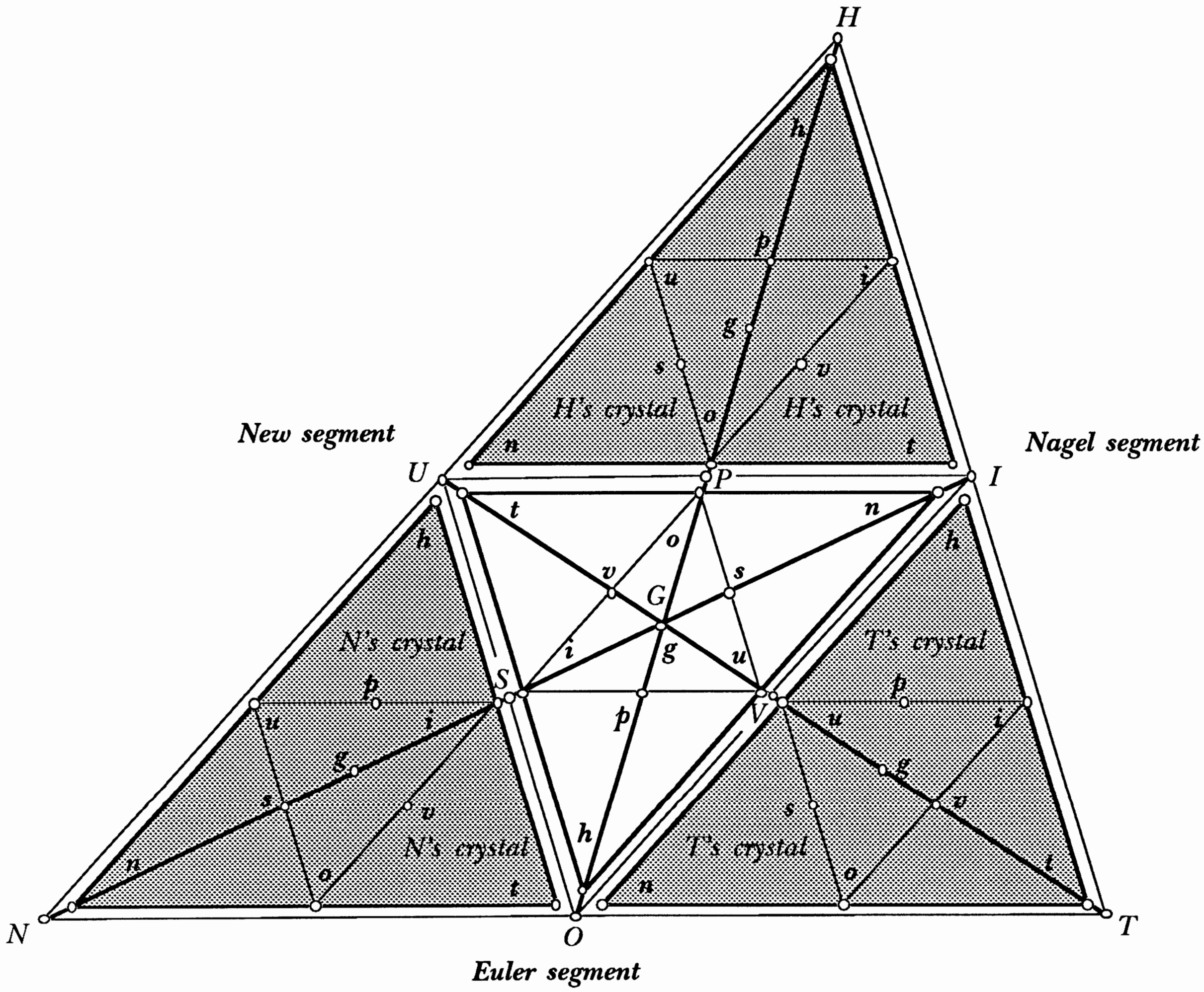
Much the same goes for analogies 3, 4, and 9. Any number of people have looked at a picture of crisscrossing medians cutting each other in their characteristic way, yet not seen the Euler and Nagel lines in it. The difference was, of course, that I was rather obsessed with the Euler/Nagel connection, so I came to the medians picture with a highly biased eye. These cases exemplify Ulam's "AS": seeing something *as* something else. They are the kinds of things that cognitive scientists interested in the deep underpinnings of creativity have to study very carefully.

Like me, Ulam was fascinated by analogies, and he reveled especially in very abstract ones, often making analogies between analogies, and carrying the game of "meta" to even higher levels. (To celebrate this trait of her husband's mind, Françoise

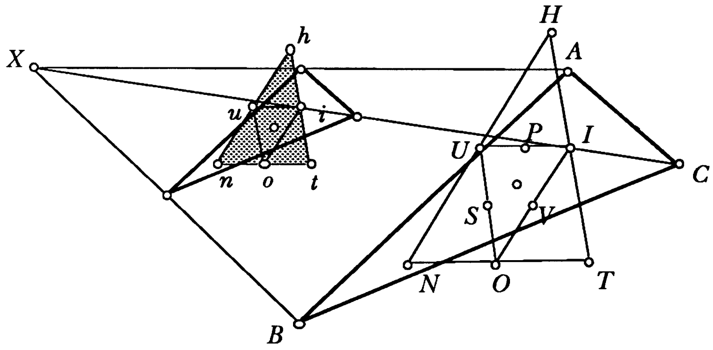
Ulam chose the title *Analogies between Analogies* for a posthumous book of his writings ranging over a vast variety of subjects, mathematical and otherwise.) Once again, I think Ulam hit the nail on the head. Mathematical thinking is permeated with analogies between analogies, although it is not often recognized as such.

It is time to wind up this lengthy and multifaceted discussion, but I could not conclude without saying that even during the writing of this essay, I found a new way of looking at the Garland Theorem, which led to a new way of looking at the hemiolic crystal, which in turn yielded two important extensions of the Garland Theorem itself. (Around and around it goes!)

What I noticed was that every line in the Garland Theorem came from the nesting of a little hemiolic crystal *hunoti,* belonging to some point's auxiliary triangle, inside a corner of the big hemiolic crystal *HUNOTI.* The various points whose auxiliary triangles were involved were, in fact, *H, N,* and *T.* I drew a picture that captured this set of nestings in a very pretty way.

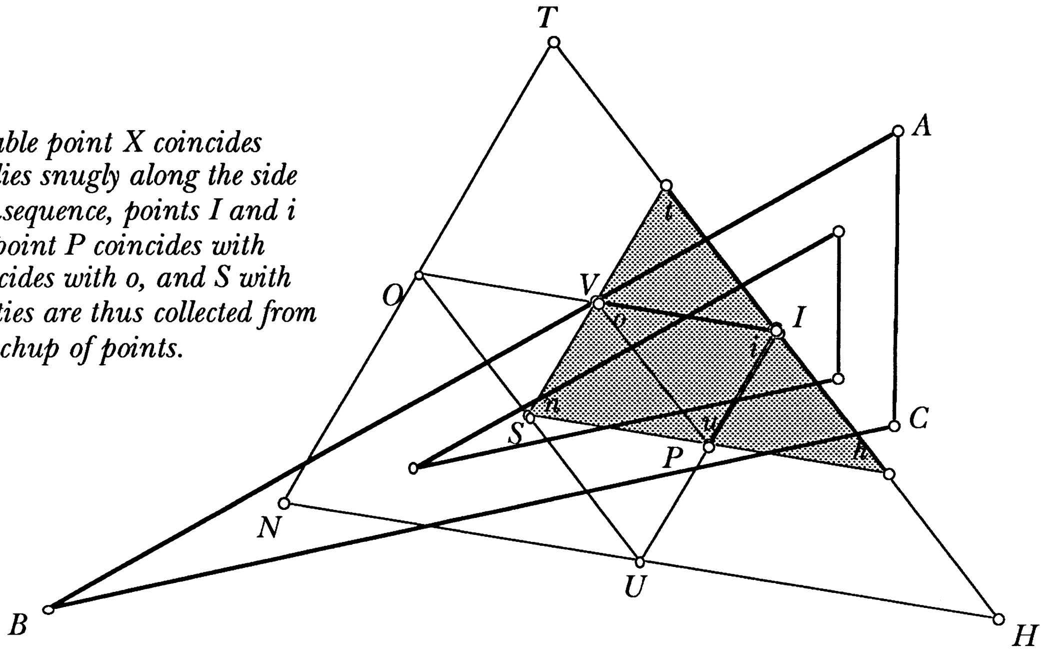


This seemed like such a rich picture that I felt I should really consider the *general* case, involving the auxiliary triangle of an *arbitrary* point *X.* In my mind, I could almost see *X* moving around and its associated little crystal following along. To find new results, all I would need to do was to slide *X* around, locating those special places where its little triangle *hunoti* fit neatly into the big triangle *HUNOTI,* giving alignments of several points at once. I could do this in my mind with some effort, but it seemed more exciting to *watch* the process, so I constructed a dynamogram.



This dynamogram allows you to find all the identities comprising the Garland Theorem, and to see why they are true. HUNOTI is the hemiolic crystal of ABC, and the shaded hunoti is the hemiolic crystal of the auxiliary triangle of point X (bold lines). As X moves around, hunoti will follow along, and in certain positions hunoti will slide into alignment with various subtriangles of HUNOTI, thus revealing hidden identities among points.

The first reward I reaped from this dynamogram was the observation that when *X* coincides with any of the "vowel points" *IOU,* its little crystal *hunoti* aligns in a different way with points of *HUNOTI* — not nesting into a corner, but nestling against a side. This simple observation instantly provided 12 new results — four coming from each vowel point.



When the variable point X coincides with I, hunoti lies snugly along the side HIT. As a consequence, points I and i are identified, point P coincides with point u, V coincides with o, and S with n. Four identities are thus collected from this simple matchup of points.

I noticed that in this "side-nestling" type of picture, there were always two points of *hunoti* sitting on the side of *HUNOTI,* exactly halfway between the chosen vowel point and its two neighboring vertices. For example, in the picture above, points *h* and *t,* flanking the vowel point *I,* do this. But if you now look back at the earlier pictures where *hunoti* is nestling in a corner of *HUNOTI,* you find that these same two points get *different* labels. For example, one finds that the spot labeled *h* in this picture (*i.e.,* the spot halfway between *H* and *I*) is labeled *i* in the fancy corner-nesting picture. This tells us that "The *I* of *H* is the *H* of *I*", or in less formal language, the incenter of the orthocenter's auxiliary triangle is the orthocenter of the incenter's auxiliary triangle. Altogether, six new identities of this sort are furnished by this observation.

A similar kind of comparison of two positions for point *X* provided three more identities involving the centroid *G,* which had up till then been curiously immune to being the subject of any garland-like identities.

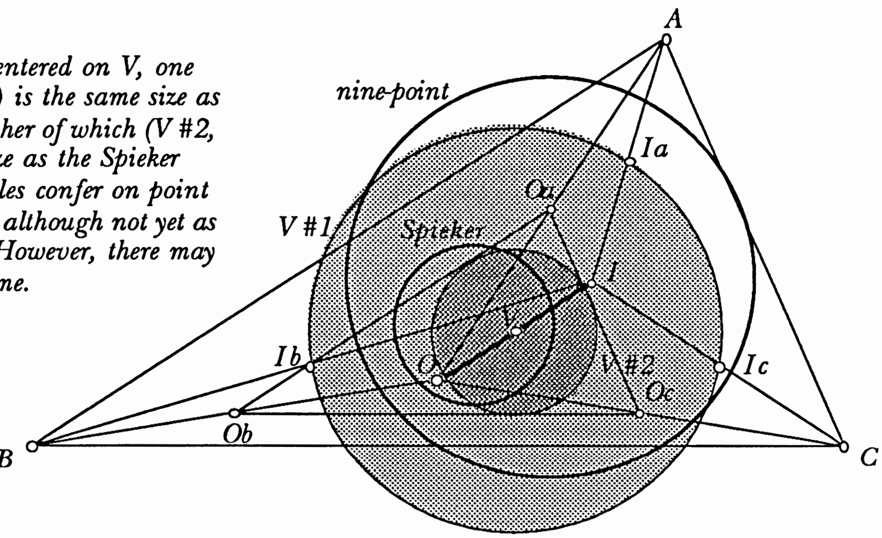
Altogether, then, I had reaped a much richer harvest than I had ever dreamed of. My full Garland Theorem could be combined with the *OVIPUS-hunoti* Theorem, and the two of them could be concisely summarized in the following schematic way:

Garland Theorem, full version. For any point X, define X's *auxiliary triangle,* denoted "X-Δ", by joining X with each of A, B, and C, and connecting the midpoints of those segments. Also let "M-Δ" stand for ABC's median triangle. The following relations then hold among the points of the various hemiolic crystals:

$$OVIPUS = hunoti\,(M\text{-}\Delta)\ \text{[meaning } O = h\,(M\text{-}\Delta),\ V = u\,(M\text{-}\Delta),\text{ etc.]};$$
$$UPIH = noth\,(H\text{-}\Delta);\ OSUN = tihn\,(N\text{-}\Delta);\ IVOT = hunt\,(T\text{-}\Delta);$$
$$SOVP = uoih\,(O\Delta);\ VIPS = oiun\,(I\text{-}\Delta);\ PUSV = iuot\,(U\text{-}\Delta);$$
$$i\,(H\text{-}\Delta) = h\,(I\text{-}\Delta);\ u(H\text{-}\Delta) = h\,(U\text{-}\Delta);$$
$$u(N\text{-}\Delta) = n\,(U\text{-}\Delta);\ o\,(N\text{-}\Delta) = n\,(O\Delta);$$
$$o\,(T\text{-}\Delta) = t(O\Delta);\ i(T\text{-}\Delta) = t\,(I\text{-}\Delta);$$
$$h\,(G\Delta) = g(H\text{-}\Delta);\ n\,(G\Delta) = g(N\text{-}\Delta);\ t\,(G\Delta) = g(T\text{-}\Delta).$$

Each of these "equations" provides a description (or set of descriptions) for some point (or set of points). For example, the equation $UPIH = noth\,(H\text{-}\Delta)$ tells us four things, one of which is that P (the nine-point center) is the circumcenter (*i.e.,* the O) of H's auxiliary triangle. This is one of the famous properties of the nine-point center. We can actually read off some interesting features of the V-point from this theorem. The equation $SOVP = uoih\,(O\Delta)$ tells us that V is the incenter of the auxiliary triangle belonging to O, and $VIPS = oiun\,(I\text{-}\Delta)$ tells us that V is the circumcenter of the auxiliary triangle belonging to I. When you look at two circles involved, you find that one of them is the same size as the nine-point circle, and the other is the same size as the Spieker circle. So my early hunch that V might have a fascinating new circle associated with it was not too far off — it's just that we have to settle for *two* circles instead of one.

There are two notable circles centered on V, one of which (V #1, lightly shaded) is the same size as the nine-point circle, and the other of which (V #2, heavily shaded) is the same size as the Spieker circle. Together, these two circles confer on point V a certain level of distinction, although not yet as impressive as that of P or S. However, there may well be more discoveries to come.



This is pretty much where matters stand, now. There is a large and symmetric batch of results that go at least some distance toward resolving the mystery of *why* there is such a tight analogy between the Nagel segment and the Euler segment, and between the Spieker circle and the nine-point circle. This set of results doesn't completely explain those mysteries, of course, but it reveals them to be merely isolated elements of one much larger complex of parallel and cyclically intertwined results.

The story of the hemiolic crystal is by no means a closed book. Indeed, I see many fascinating avenues to explore. One has to do with the fact that you can take an unlabeled picture of a hemiolic crystal and legally label its points in four different ways. (It might seem that there should be *six* ways to label it, but as it turns out, there is a constraint: the UT median cannot be as long as the Euler median OH. This limits the possible labelings to four.) Thus one has *HUNOTI, HITONU, NUHITO,* and *NOTIHU.* All the "cousin" crystals belong to cousin triangles ABC, $A'B'C'$, $A''B''C''$, and $A'''B'''C'''$ — and so the obvious question is: How are all four cousin triangles related to one another?

But the biggest remaining mystery for me concerns the meanings of the points U and T. Although they play beautifully symmetric roles in the Garland Theorem, I have so far been unable to find any concise and catchy characterizations for them on their own. Just what *is* the T-point? What *is* the U-point? Mysteries beckon, mysteries call, mysteries ever lure me on...

A few weeks ago, flush with excitement about the earliest of these discoveries, I penned a letter to the great geometer H. S. M. Coxeter, hoping to see if my ideas were new and of merit. After briefly describing my new-found passion for geometry and recounting my discovery of the hemiolic crystal and its properties, I concluded with the following lines: "I will never be quite the same, after having drunk so deeply from the infinite well of geometry. My life is in some central way forever changed, thanks to the mysteries and beauties of triangles and circles." And so it is.